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LECTURES ON PARTIAL DIFFERENTIAL EQUATIONS

I. G. PETROVSKY

Translated from the Russian by

A. SHENITZER

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FOREWORD

Professor Petrovsky is an outstanding and leading representative of the great mathematical tradition and of the most impressive active mathematical life of Russia. Great progress is due to him in the field of partial differential equations. It will, therefore, be highly welcome to English speaking students that Petrovsky's masterly lectures on this important subject are now being made accessible through the present translation from the Russian original.

R. COURANT

TRANSLATOR'S NOTE

The work of an eminent mathematician, Petrovsky's *Partial Differential Equations,* has many qualities which should make it popular with graduate students desiring to get acquainted with partial differential equations, a field of mathematics of great intrinsic beauty and of vast importance in applications.

The presentation of the material is rigorous and at the same time transparent and readable. Petrovsky recognizes the value of heuristic and physical considerations but at the same time maintains high mathematical standards. His comments on and discussion of various aspects of the problems under consideration cannot but deepen the students' understanding and appreciation of these problems.

A. SHENITZER

PREFACE

The present book grew out of a series of lectures which I presented a number of times to students of mathematics at the department of mathematics and mechanics of the Moscow State University. These lectures were to some extent supplemented before being printed.

In §§ 24 and 35, integral equations are used in presenting the material covered in §§ 23 and 31 through 33. Consequently, the material exclusive of §§ 24 and 35 (as well as 34) is self-contained.

In the preparation of this book I was given a great deal of assistance. K. S. Kusmin prepared notes of my lectures. Z. Ya. Shapiro was particularly helpful in that she edited the manuscript and wrote §§ 22 through 25 and portions of other sections. Without her help the publication of this book would have been considerably delayed. A. D. Myshkis and M. I. Vishik read the manuscript and made a number of very useful remarks. A. D. Myshkis also wrote §§ 34, 35, and part of § 4. B. M. Levitan wrote para. 3 of § 26. I wish to express my deep gratitude to all of them.

I. PETROVSKY

9 *April* 1950

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CHAPTER I

INTRODUCTION. CLASSIFICATION OF EQUATIONS

§ 1. Definitions. Examples

1. An equation containing partial derivatives of unknown functions $u_1, u_2, \ldots, u_N$ is said to be of the nth order if it contains at least one partial derivative of the nth order and no partial derivatives of order higher than n. By the order of a system of equations containing partial derivatives we mean the order of the highest order equation of the system.

A partial differential equation is called linear if it is linear in the unknown functions and in their derivatives; it is called quasi-linear if it is linear in the highest order derivatives of the unknown functions. Thus, for instance, the equation

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+u^2=0$$

is quasi-linear of the second order with respect to the unknown function u. The equation

$$\frac{\partial^2 u}{\partial x^2}+a(x,y)\frac{\partial^2 u}{\partial x^2}=2u$$

is linear of the second order with respect to u. The equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=u$$

is neither linear nor quasi-linear with respect to u.

By a solution of an equation containing partial derivatives we mean a system of functions which, when put in the equation in place of the unknown functions, turns the equation into an identity in the independent variables. A solution of a system of equations is defined in an analogous manner.

We shall be primarily interested in linear equations of the second order in one unknown function. The following are examples of equations of this type:

(1) $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}$$ the 'heat equation',

(2) $$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}$$ the 'wave equation',

(3) ✓ $$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}=0$$ the 'Laplace equation'.

Many problems in physics reduce themselves to partial differential equations, in particular, to the partial differential equations listed above.

2. *Example* 1. *The heat equation.* This equation describes the flow of heat. Its derivation is based on the following physical law (Newton's law):

Let the temperature of a body at a point (x_1, x_2, x_3) in the body at the time t be given by a continuous and differentiable function $u(t, x_1, x_2, x_3)$. Then the formula

$$\Delta Q \approx -k(x_1, x_2, x_3)\,\frac{\partial u(t, x_1, x_2, x_3)}{\partial n}\,\Delta S\,\Delta t$$

gives the amount of heat flowing through a small area ΔS in the body during the time Δt to within terms of order higher than the order of the product $\Delta S\,\Delta t$. Here $\partial u/\partial n$ denotes the normal derivative in the direction of the flow of heat evaluated at an arbitrary point (x_1, x_2, x_3) of ΔS. The positive coefficient $k(x_1, x_2, x_3)$ is called the coefficient of *inner thermal conductivity* at the point (x_1, x_2, x_3).

If the element of area under consideration forms part of the boundary of the body, then the same law in somewhat different form holds. Again, let $u(t, x_1, x_2, x_3)$ denote the temperature of the body G at the point (x_1, x_2, x_3) and let $u_1(t, x_1, x_2, x_3)$ denote the temperature at an arbitrary point (x_1, x_2, x_3) outside the body. Then, the amount of heat entering the body across the element of area ΔS in the time interval Δt is given, to within higher order terms, by the expression

$$\Delta Q \approx k_1(x_1, x_2, x_3)\,[u_1(t, x_1, x_2, x_3) - u(t, x_1, x_2, x_3)]\,\Delta S\,\Delta t.$$

Here (x_1, x_2, x_3) denotes an arbitrary point of the element of area, and the values of the functions u_1 and u at that point are limiting values obtained by approaching the point from the outside and from the inside of the body, respectively. The coefficient $k_1(x_1, x_2, x_3)$ is referred to as the coefficient of *outer thermal conductivity* relative to the given medium.

We shall assume that the body is isotropic with respect to heat conduction, i.e. $k(x_1, x_2, x_3)$ is independent of the orientation of the element of area. We shall also assume that the function $k(x_1, x_2, x_3)$ has continuous partial derivatives with respect to x_1, x_2 and x_3.

For the purpose of our derivation we imagine in the interior of the considered region G a cube D defined by the inequalities

$$x_i \leqslant \xi_i \leqslant x_i + \Delta x_i.$$

Here (x_1, x_2, x_3) denotes a point in the interior of G, (ξ_1, ξ_2, ξ_3) denotes a point of the cube D, and $\Delta x_1 = \Delta x_2 = \Delta x_3$ are small increments of the coordinates. We assume that the function $u(t, x_1, x_2, x_3)$ is twice continuously differentiable. We now compute the increase in the amount of heat inside our cube over the time interval Δt. By Newton's law, the amount of heat flowing into the cube across the face lying in the plane $\xi_1 = x_1$ is given, to within terms of higher order, by

$$\Delta Q' \approx -k(x_1, x_2, x_3)\frac{\partial u(t, x_1, x_2, x_3)}{\partial x_1}\Delta t\,\Delta x_2\,\Delta x_3.$$

Similarly, for the parallel face $\xi_1 = x_1 + \Delta x_1$ we have

$$\Delta Q'' \approx k(x_1 + \Delta x_1, x_2, x_3)\frac{\partial u(t, x_1 + \Delta x_1, x_2, x_3)}{\partial x_1}\Delta t\,\Delta x_2\,\Delta x_3,$$

and, according to the mean-value theorem,

$$\Delta Q' + \Delta Q'' \approx \frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right)\Delta t\,\Delta x_1\,\Delta x_2\,\Delta x_3.$$

Carrying out analogous computations for the remaining faces of the cube, we find that the total increase in the amount of heat in the cube D in time Δt is given by

$$dQ \approx \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[k(x_1, x_2, x_3)\frac{\partial u(t, x_1, x_2, x_3)}{\partial x_i}\right]\Delta t\,\Delta x_1\,\Delta x_2\,\Delta x_3. \quad (1, 1)$$

On the other hand, this amount of heat can be estimated by considering the increase in temperature in the cube in time Δt:

$$\begin{aligned} dQ &\approx [u(t + \Delta t, x_1, x_2, x_3) \\ &\qquad - u(t, x_1, x_2, x_3)]\, c(x_1, x_2, x_3)\,\rho(x_1, x_2, x_3)\,\Delta x_1\,\Delta x_2\,\Delta x_3 \\ &\approx \frac{\partial u(t, x_1, x_2, x_3)}{\partial t}\, c(x_1, x_2, x_3)\,\rho(x_1, x_2, x_3)\,\Delta t\,\Delta x_1\,\Delta x_2\,\Delta x_3. \quad (2, 1) \end{aligned}$$

Here $\rho(x_1, x_2, x_3)$ denotes the density of the body at the point (x_1, x_2, x_3) and $c(x_1, x_2, x_3)$ its specific heat at that point.† Dividing (1, 1) and (2, 1) by $\Delta t \Delta x_1 \Delta x_2 \Delta x_3$, equating the results of division, and passing to the limit by letting $\Delta t \to 0$, $\Delta x_i \to 0$, we get an exact equality at the point (x_1, x_2, x_3) at time t:

$$c\rho \frac{\partial u}{\partial t} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(k \frac{\partial u}{\partial x_i}\right). \tag{3, 1}$$

This equation is called the *heat equation* for a non-homogeneous isotropic body. If the body is homogeneous, then

$$k(x_1, x_2, x_3) = \text{const.}, \quad c(x_1, x_2, x_3) = \text{const.}, \quad \rho(x_1, x_2, x_3) = \text{const.},$$

and equation (3, 1) becomes

$$\frac{c\rho}{k} \frac{\partial u}{\partial t} = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}. \tag{4, 1}$$

Replacing $(k/c\rho)\, t$ with t' and denoting t' by t, we obtain (4, 1) in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}. \tag{5, 1}$$

Equations (3, 1) and (5, 1) have many solutions. In order to single out one of the many solutions of these equations, it is necessary to prescribe supplementary conditions which are the counterpart of initial conditions as prescribed in ordinary differential equations. Such supplementary conditions are most often so-called *boundary conditions*, i.e. conditions prescribed on the boundary of the region G of the space (x_1, x_2, x_3) in which we are to find a solution of an equation with partial derivatives, and *initial conditions* which refer to some definite moment in time. It is clear from the physical point of view that the knowledge of the temperature of the body at some moment in time and of the distribution of heat at the boundary of the body should completely determine the temperature of the body at any later moment in time. Likewise, it is clear that the distribution of heat can be prescribed in a variety of ways. If the region G coincides with all of space, then a bounded solution of the heat equation for $t > t_0$ is uniquely determined by prescribing initial conditions only, namely, the value of the function $u(t, x_1, x_2, x_3)$ at $t = t_0$. In case of a bounded domain G one could, for example, prescribe the temperature at each point of the body at $t = t_0$ and, in addition, the temperature at the boundary points of the body for all $t > t_0$. It turns out that these conditions suffice to determine a unique, bounded solution for $t > t_0$ and $(x_1, x_2, x_3) \in G$.

---

† The value assumed by some physical characteristic of a body at a definite point P is always understood to be a limiting value. Namely, we consider a sequence of cubes with centers at the point P and with sides whose length approaches zero. For each cube we take the ratio of the relevant quantity to the volume of the cube. Then, for the value at P of the physical characteristic in question, we take the limit of the sequence of ratios as the length of the sides of the cubes approaches zero. Thus, in the case of density at a point P the relevant quantity is the mass of each cube with center at P. In the case of surface density of a thin plate the relevant quantity is the mass of a small square about the point in question. In case of a rod the relevant quantity is the mass of a segment of the rod with center at the point in question. The definitions of specific heat, conductivity, etc., at a point are arrived at in an entirely analogous manner.

Instead of prescribing $u(t, x_1, x_2, x_3)$ at the boundary of G for $t > t_0$, we could prescribe the values of the outer normal derivative $\partial u/\partial n$ of the solution function u. This condition also determines a unique solution of the heat equation. We arrive at this formulation of the problem if we study the temperature in the interior of G under the assumption that we know at all times the amount of heat which flows in time Δt from outside the body across each small surface element ΔS of the boundary of the body. This quantity must be equal to the amount of heat transmitted from the surface of the body to its interior. If this were not so, heat would keep on accumulating at the boundary of G and, since the mass of the boundary of the body is equal to zero, the temperature at the boundary would tend towards infinity. According to Newton's law, the amount of heat mentioned is equal to

$$k(\partial u/\partial n)\, \Delta S\, \Delta t.$$

Here $k > 0$ denotes the coefficient of thermal conductivity at the boundary point under consideration. Knowing the law of heat output at each boundary point of the region G, we could find the value of $\partial u/\partial n$. In particular, if the body does not give up heat at the boundary, $\partial u/\partial n = 0$ at the boundary.

Finally, it is possible to prescribe as boundary conditions for $t \geqslant t_0$ the values

$$k(\partial u/\partial n) + k_1 u.$$

Here k_1 is the coefficient of outer thermal conductivity and k the coefficient of inner thermal conductivity. The values of these coefficients are assumed to be known. We arrive at a mathematical problem of this kind if we study the temperature in the interior of the body G under the assumption that we know the temperature u_1 of the body surrounding G. Then, according to Newton's law, we shall find that:

(1) The amount of heat flowing from the surrounding medium across a surface element ΔS in the time interval $(t, t+\Delta t)$ equals, apart from terms of higher order,

$$k_1(u_1-u)\,\Delta S\,\Delta t \quad (k_1>0).$$

(2) The amount of heat transmitted during that time from the surface element ΔS to the interior of the body equals, to within terms of higher order,

$$k(\partial u/\partial n)\,\Delta S\,\Delta t \quad (k>0),$$

so that, in view of the equality of the two amounts,

$$k_1 u + k(\partial u/\partial n) = k_1 u_1.$$

In particular, if $u_1 \equiv 0$, this condition becomes

$$k(\partial u/\partial n) + k_1 u = 0.$$

Let us now assume that the temperature at each interior point (x_1, x_2, x_3) of the body does not vary with increasing time. Then $\partial u/\partial t = 0$ and equations (3, 1) and (5, 1) go over respectively into

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(k\frac{\partial u}{\partial x_i}\right) = 0, \quad \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = 0. \tag{6, 1}$$

To determine $u(x_1, x_2, x_3)$ it is not necessary, in this case, to prescribe initial conditions. It suffices to prescribe boundary conditions which we shall regard as being independent of time. The physical picture in this case is as follows. If the boundary conditions do not change in time, then, regardless of the prescribed initial conditions, the temperature $u(t, x_1, x_2, x_3)$ at each point (x_1, x_2, x_3) of the body goes to a definite limiting value $u(x_1, x_2, x_3)$. The limit function $u(x_1, x_2, x_3)$ satisfies the steady-state equations (6, 1) as well as the boundary conditions assumed to be independent of time.

The problem of determining a function satisfying either one of the equations (6, 1) as well as prescribed boundary conditions is called the *Dirichlet problem* or *the first boundary-value problem.*

The problem of heat conduction in 3-space has its counterparts in 2-space and in 1-space. Thus, consider a rod of which we assume that its temperature is the same for all points of a given cross-section and that no heat exchange takes place between the surrounding medium and the rod across the lateral surface of the rod. Then the temperature u of the rod will be a function of the time t and of one space coordinate x only. With a proper choice of units, the equation satisfied by the function $u(t,x)$ will, in this case, be of the form

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}. \tag{7, 1}$$

If the temperature $u(t,x_1,x_2,x_3)$ of a three-dimensional body were to depend on only one space coordinate, $x_1=x$, say, then u would also satisfy equation (7, 1). This would take place if the temperature of the body were the same for all points of the plane $x=\text{const.}$, for an arbitrary value of that constant. Likewise, the study of conduction of heat in a thin plate leads to the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}. \tag{8, 1}$$

3. *Example* 2. *Equation of equilibrium and vibration of a membrane.* We shall consider a membrane which when at rest is in the (x_1,x_2) plane. By a membrane we mean a stretched film which resists stretching but does not resist bending, i.e. changes of form which do not alter the area of an arbitrary piece of the membrane. The work of the outside force which produces the change in form is proportional to the change itself. The positive coefficient of proportionalty T does not depend on the form or position of the membrane element. We call T the *tension* of the membrane.

The following considerations clarify the term *tension*.

If we cut the membrane along an arbitrary straight line segment l and consider the part of the membrane on one side of that segment, then, as will be shown, the action on it of the other part of the membrane is the same as the action of a force Tl applied at l and perpendicular to it.

We first show that the force T_l acting on l is perpendicular to l. To this end we consider a small piece of the membrane in the form of a plane parallelogram (cf. fig. 1). We denote one side of this parallelogram by l. Keeping the side of the parallelogram opposite to l fixed, we deform the membrane so that the side l remains on one line. At the same time the remaining two sides of the parallelogram remain parallel to each other and its area will stay fixed. We now replace the action of the membrane on our parallelogram by forces applied at its sides. Since the change in area of the parallelogram is zero, the work of the sum of the forces is also zero. Since the side AD (opposite to l) is fixed, the work of the force applied at AD is zero. Again, since our parallelogram is small, the forces applied at the parallel sides AD and BC are almost equal and almost oppositely directed so that the work of these forces is zero to within terms of order higher than that of the work of the force T_l. Hence, the work

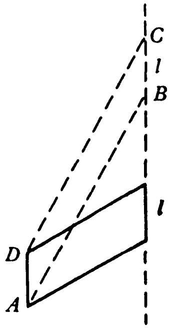

Figure 1

of the force applied at the side l is also equal to zero. Consequently, the force T_l must be perpendicular to l.

We now show that $T_l = Tl$. For this purpose we consider a piece of the membrane in the form of a rectangle. Let l be one side of this rectangle. If we now deform the membrane so that the side l is moved a distance λ parallel to itself and the position lines of the remaining sides stay fixed, then the area of our rectangle is increased by λl. It follows from the definition of tension that the work associated with this deformation of our rectangle is $\lambda l T$. If, on the other hand, we again replace the action of the membrane on our rectangle by forces applied at its sides and denote by T_l the force applied at the side l, we find that the work considered is equal to $T_l \lambda$. Equating the two expressions for the work done in deforming our rectangle, we get

$$T_l = Tl.$$

It is, therefore, in order to call T the tension of the membrane.

Now let the membrane, *when at rest*, coincide with a plane region G in the (x_1, x_2) plane. Let L denote the boundary of G.

We assume that each element $\Delta x_1 \Delta x_2$ (see fig. 2) of the membrane is acted on by a force given, to within infinitesimals of higher order, by $f(x_1, x_2)\Delta x_1 \Delta x_2$ and having the direction of the normal to the plane (x_1, x_2). The membrane will bend under the action of this force and will assume the form of some surface

$$u = u(x_1, x_2),$$

say. The u-axis is assumed to be perpendicular to the plane (x_1, x_2).

We shall derive the equation satisfied by the function $u(x_1, x_2)$ under the following restrictions. First, when in equilibrium, the surface of the membrane is bent little, i.e. it approximates a portion of the plane. This means that the derivatives $\partial u/\partial x_1$ and $\partial u/\partial x_2$ are small and that we may disregard powers of these derivatives higher than the first. Secondly, under the action of the force $f(x_1, x_2)$, the points of the membrane move only in directions normal to the plane (x_1, x_2) so that their coordinates (x_1, x_2) do not vary with the motion.

The derivation of the equation is based on one of the fundamental principles of mechanics, the principle of virtual work. This principle asserts that if a system is in a state of equilibrium, then in any small displacement of the system compatible with the constraints imposed on the system† the total work done by all the forces acting on the system is zero.

We first compute the work done by the forces acting on the element $\Delta x_1 \Delta x_2$ (fig. 2) when this element is displaced from its initial position in the (x_1, x_2) plane to a position described by the equation $u = u(x_1, x_2)$. The forces acting on this element are as follows: the outwardly directed (with respect to our element) force of tension and the force $f\Delta x_1 \Delta x_2$. The work of the force of tension is proportional to the increase in area accompanying the displacement of the membrane. The area of the surface element ΔS whose projection is the square $\Delta x_1 \Delta x_2$ is approximately equal to

---

† Cf. S. Banach, 'Mechanics', *Monogr. mat.* vol. XXIV (1951). G. K. Suslov, *Theoretical Mechanics*, Gostekhizdat (1944) (in Russian).

$$(1 + u'^2_{x_1} + u'^2_{x_2})^{\frac{1}{2}} \Delta x_1 \Delta x_2.$$

In the state of equilibrium this element coincides with the square $\Delta x_1 \Delta x_2$. Consequently, the increase in area is equal to

$$(1 + u'^2_{x_1} + u'^2_{x_2})^{\frac{1}{2}} \Delta x_1 \Delta x_2 - \Delta x_1 \Delta x_2.$$

It follows that the work of the (outwardly directed) forces of tension is given by

$$T[(1 + u'^2_{x_1} + u'^2_{x_2})^{\frac{1}{2}} - 1] \Delta x_1 \Delta x_2.$$

If we now expand the term in brackets in a power series in the variables u'_{x_1} and u'_{x_2} and disregard higher order terms in the expansion, then our expression reduces to

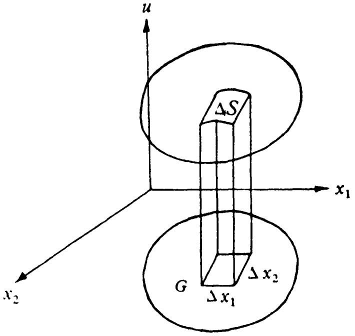

Figure 2

$$\tfrac{1}{2}T(u'^2_{x_1} + u'^2_{x_2}) \Delta x_1 \Delta x_2.$$

As for the work of the force $f\Delta x_1 \Delta x_2$, it is simply

$$fu\Delta x_1 \Delta x_2.$$

Since the work of the inner elastic forces in the element $\Delta x_1 \Delta x_2$ must be opposite in sign to the work of the outer forces of tension acting on this element and equal to it in absolute value, the work done on the membrane by all inner forces acting on the membrane and by the force f in a displacement of the membrane from the position of rest to some position $u(x_1, x_2)$ is given by

$$A(u) = \iint_G [-\tfrac{1}{2}T(u'^2_{x_1} + u'^2_{x_2}) + fu]\, dx_1 dx_2. \qquad (9, 1)†$$

---

† The integral (9, 1) is equal to the potential energy of the membrane in the position of equilibrium. We could therefore say that our derivation is based on the fact that in a position of equilibrium the potential energy of any mechanical system takes on a minimal value.

We shall now effect an admissible displacement of the membrane, i.e. we shall add a certain function $\delta u(x_1, x_2)$ to the function $u(x_1, x_2)$. In this displacement the work of all acting forces equals the variation of the integral (9, 1). This variation is easily seen to be equal to

$$\delta A = A(u+\delta u) - A(u) \approx \iint_G [-T(u'_{x_1}\delta u'_{x_1} + u'_{x_2}\delta u'_{x_2}) + f\delta u]\, dx_1 dx_2. \tag{10, 1}$$

According to the principle of virtual work this variation must equal zero.

Integrating by parts we obtain the well-known formula

$$\iint_G (u'_{x_1}\delta u'_{x_1} + u'_{x_2}\delta u'_{x_2})\, dx_1\, dx_2$$
$$= \int_L \frac{\partial u}{\partial n}\delta u\, ds - \iint_G (u''_{x_1^2} + u''_{x_2^2})\,\delta u\, dx_1 dx_2,$$

so that $$\delta A = \int_L -T\frac{\partial u}{\partial n}\delta u\, ds + \iint_G (T\Delta u + f)\,\delta u\, dx_1 dx_2. \tag{11, 1}$$

Here Δu denotes the sum

$$\partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2.$$

We repeat that δu denotes a virtual displacement, i.e. one compatible with the constraints imposed on the membrane. Such constraints are usually imposed on the boundary of the membrane, so that $\delta u(x_1, x_2)$ is an arbitrary function continuous at all interior points of the membrane. Consequently, $\delta A = 0$ implies that in the position of equilibrium the function $u(x_1, x_2)$ satisfies the equation

$$T\Delta u + f = 0, \tag{12, 1}$$

at all interior points. Equation (12, 1) is called the *Poisson equation.*

The constraints take the form of various boundary conditions. We consider below the most frequent cases.

I. *The fastened membrane.* Let the edge of the membrane be fastened to some space curve whose projection is the curve L. Let $x_1 = x_1(s)$, $x_2 = x_2(s)$ be the parametric equations of L. Then the membrane is assumed to go through some curve $x_1 = x_1(s)$, $x_2 = x_2(s)$; $u = \phi(s)$. With this single restriction on δu we have $\delta u = 0$ on L. Also, the line integral in (11, 1) will vanish.

The resulting problem, which consists in finding a solution of Poisson's equation satisfying the boundary condition $u = \phi(s)$ on L, is called the *Dirichlet problem* for this equation.

If $f = 0$, Poisson's equation reduces to Laplace's equation with which we are already familiar.

II. *The free membrane.* If we do not impose any restrictions on the position of the membrane, then its boundary is free to move along the (vertical) side of the cylinder with base boundary L. In that case δu is arbitrary both inside G and on the boundary of G, and we obtain for equation (12, 1) the condition

$$\partial u/\partial n = 0$$

on L.

III. The following is a frequent case. In addition to the force $f\Delta x_1 \Delta x_2$ acting on the interior points of the membrane, a vertical force with linear density f_1 is applied at the boundary so that there is a force $f_1 \Delta s$ acting on a boundary element of length Δs. If we inquire after the position of equilibrium of the membrane under these conditions, then we must add the term $\int_L f_1 u\, ds$ to the integral in (9, 1) and the term $\int_L f_1 \delta u\, ds$ to the integral in (10, 1).

In this case the line integral in (11, 1) is of the form

$$\int_L [-T(\partial u/\partial n) + f_1]\, \delta u\, ds,$$

and the boundary condition for the Poisson equation is

$$T(\partial u/\partial n) - f_1 = 0$$

on L. This problem is known as the *second boundary-value problem* (*the Neumann problem*).

IV. Sometimes one considers the case of an 'angularly fastened membrane', i.e. the case in which the force is

$$f_1(s) = ku(s).$$

In that case the boundary condition for the Poisson equation is

$$T(\partial u/\partial n) - ku = 0.\dagger$$

We shall now derive the *equation of motion of a membrane*. We shall consider only transverse vibrations of small amplitude. Trans-

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† Clearly, the boundary conditions II, III and IV are all equilibrium conditions for the forces acting on an element Δs of the boundary of the membrane. In fact, the force of tension exerted by the interior of the membrane on an element Δs of its boundary lies in a tangent plane and its direction is normal to the boundary. Its projection $T(\partial u/\partial n)ds$ in the plane (x_1, x_2) must be zero (free boundary) or in equilibrium with the forces acting on the boundary.

verse vibrations are vibrations in a direction perpendicular to the (x_1, x_2) plane. Clearly, in the case of transverse vibrations the only coordinate of a fixed point in the membrane which varies with time is its u coordinate, $u = u(t, x_1, x_2)$. The small amplitude of the vibrations is synonymous with the smallness of u, $\partial u/\partial x_1$ and $\partial u/\partial x_2$. We made use of such smallness assumptions in deriving (9, 1).

The velocity of a point of the membrane with coordinates (x_1, x_2) is given by $\partial u(t, x_1, x_2)/\partial t$ and its acceleration by $\partial^2 u(t, x_1, x_2)/\partial t^2$. To obtain the equation of motion of the membrane we replace the outer force $f\Delta x_1 \Delta x_2$ by the product of the mass of the element and its acceleration. The sign of this product is to be the negative of the sign of the force (D'Alembert's principle). The mass of an element $\Delta x_1 \Delta x_2$ is equal to $\rho(x_1, x_2)\Delta x_1 \Delta x_2$, where $\rho(x_1, x_2)$ stands for the surface density at the point (x_1, x_2). Hence, to obtain the equation for the transverse vibrations of the membrane we must replace the second term in (12, 1) by $-\rho \partial^2 u/\partial t^2$,

$$T\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) - \rho \frac{\partial^2 u}{\partial t^2} = 0. \tag{13, 1}$$

Here the possible boundary conditions are the same as in the case of equation (12, 1), except for the fact that the functions prescribed on the boundary can now be functions of time. The most frequently encountered problems are the problem of the membrane whose boundary is fastened along the curve L, $u(t, x_1, x_2) = 0$ on L, and the problem of the free membrane, $\partial u(t, x_1, x_2)/\partial n = 0$ on L.

Just as in the case of the heat equation it is clear from the physical point of view that the boundary conditions cannot by themselves determine uniquely the motion of a membrane; the reason being that this motion depends in an essential manner on the initial position and on the initial velocity. In fact, it will be shown later that a solution of (13, 1) is determined uniquely by prescribing initial conditions

$$\left.\begin{aligned} u(t_0, x_1, x_2) &= \phi_0(x_1, x_2) \\ u_t'(t_0, x_1, x_2) &= \phi_1(x_1, x_2) \end{aligned}\right\} \quad (x_1, x_2) \in G, \tag{14, 1}$$

and boundary conditions of any of the types considered.

It is of theoretical interest to consider the case of the so-called infinite membrane, i.e. the vibrations of the whole plane (x_1, x_2) satisfying equation (13, 1). In that case, as will be shown later, a unique solution of (13, 1) is determined by prescribing initial conditions only. In case of a finite membrane initial conditions determine a unique solution not for all t but for a finite time interval $(-T, T)$. The time interval $(-T, T)$ varies with the point (x_1, x_2), and its magnitude decreases as we approach the boundary of the region G.

If ρ is constant, then a suitable change of coordinates will reduce (13, 1) to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}. \tag{15, 1}$$

Consideration of the vibrations of a three-dimensional, elastic body leads, by means of similar considerations, to the equation of motion

$$\rho \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right), \tag{16, 1}$$

for the function $u(t, x_1, x_2, x_3)$.

The equation describing the vibration of a gas is of the same form. In that case we may interpret the function $u(t, x_1, x_2, x_3)$ as the deviation from the normal pressure at the point (x_1, x_2, x_3) at time t. In the one-dimensional case (the vibration of a string, or of a gas in a pipe) the function u satisfies the equation

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}. \tag{17, 1}$$

This equation is called the *equation of the vibrating string*. Here $\rho(x)$ denotes the linear density at the point x and T the tension in the string. The initial and boundary conditions for (16, 1) and (17, 1) are entirely analogous to the initial and boundary conditions for (13, 1).

We emphasize that we obtain equations (13, 1), (16, 1) and (17, 1) only if we regard $(\partial u/\partial x_i)^4$ as being negligible in comparison with $(\partial u/\partial x_i)^2$. If we do not make this assumption (i.e. if we do not assume that the vibrations are small), the equations of motion become far more complicated. They also become non-linear.

Remark 1. If t is also regarded as a space coordinate, then the function $u(t, x_1, x_2)$, which describes the vibration of the membrane, will be defined in a cylinder C with generators going through the boundary of the region G (the projection of the membrane) and having the direction of the t axis. The problem considered above consists in determining the values of this function inside the cylinder given the values of $u(t_0, x_1, x_2)$ and $u_t'(t_0, x_1, x_2)$ for (x_1, x_2) in G and given certain conditions on the side of the cylinder C. If this point of view is adopted, then both the initial and the boundary conditions become boundary conditions prescribed on the boundary of the cylinder C.

Remark 2. We recall that the expression

$$(\partial^2 u/\partial x_1^2)+(\partial^2 u/\partial x_2^2) \qquad (18,1)_1$$

enters into the heat equation, and the expression

$$(\partial^2 u/\partial x_1^2)+(\partial^2 u/\partial x_2^2)+(\partial^2 u/\partial x_3^2) \qquad (18,1)_2$$

enters into the equation of vibrations in an isotropic medium. This is always the case in linear equations of the second order for an isotropic two- or three-dimensional medium; the reason being that the expressions (18, 1), sometimes referred to as Laplacians, are, apart from a constant multiplier, the only linear combinations of the second-order partial derivatives of u which are invariant under orthogonal transformations (i.e. under a rotation of the orthogonal coordinates in two- or three-dimensional space).

§ 2. The Cauchy problem. The Cauchy-Kowalewski theorem

1. *Formulation of the Cauchy problem.* Let there be given the following system of equations in the partial derivatives of unknown functions $u_1, \ldots, u_N$ with respect to the independent variables $t, x_1, \ldots, x_n$:

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = F_i\left(t, x_1, \ldots, x_n, u_1, \ldots, u_N, \ldots, \frac{\partial^k u_j}{\partial t^{k_0}\partial x_1^{k_1}\ldots\partial x_n^{k_n}}, \ldots\right) \qquad (1,2)$$

$$(i,j=1,2,\ldots,N; \quad k_0+k_1+\ldots+k_n=k\leqslant n_j; \quad k_0<n_j).$$

It is clear from our equations that for each of the unknown functions u_i there exists a number n_i which gives the order of the highest order derivatives of u_i appearing in the system. The independent variable t is seen to be distinguished in two respects. First, the derivative $\partial^{n_i}u_i/\partial t^{n_i}$ must appear among the derivatives of the highest order n_i of the function u_i $(i=1,2,\ldots,N)$ and, secondly, the system is solved for these derivatives. In physical problems time occupies this distinguished position and the other variables $x_1, \ldots, x_n$ denote space coordinates. The number of equations is equal to the number of unknown functions.

For some value $t=t_0$ of t we prescribe the values ('initial values') of the unknown functions and of their derivatives with respect to t of orders up to n_i-1. In other words, for $t=t_0$

$$\partial^k u_i/\partial t^k = \phi_i^{(k)}(x_1, x_2, \ldots, x_n) \quad (k=0,1,2,\ldots,n_i-1). \qquad (2,2)$$

All functions $\phi_i^{(k)}(x_1, x_2, \ldots, x_n)$ are prescribed in the same region G of the space $(x_1, x_2, \ldots, x_n)$. By the derivative of order zero of the function u_i we mean the function u_i itself.

The Cauchy problem consists in finding a solution of the system (1, 2) *satisfying the initial conditions* (2, 2).

The problem (mentioned in § 1) of determining the vibrations of a homogeneous membrane whose motion is to satisfy given initial conditions is a special case of the Cauchy problem. Its analytic formulation is as follows: to find a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

satisfying the following initial conditions:

$$u(t_0, x_1, x_2) = \phi^{(0)}(x_1, x_2) \quad \text{(initial displacement)},$$

$$u_t'(t_0, x_1, x_2) = \phi^{(1)}(x_1, x_2) \quad \text{(initial velocity)}.$$

If $N = 1$, $n_1 = 1$, $n = 0$, the Cauchy problem is reduced to the problem of finding a solution $u(t)$ of the ordinary differential equation

$$du/dt = F(t, u),$$

such that $u(t_0) = u_0$. This problem is studied in detail in the theory of ordinary differential equations.

2. A function $F(z_1, z_2, \ldots, z_m)$ of m complex variables is called analytic in a neighborhood of a point $z_1^0, z_2^0, \ldots, z_m^0$ if it is representable as a power series

$$F(z_1, z_2, \ldots, z_m) = \sum_{k_1 k_2 \ldots k_m} A_{k_1 k_2 \ldots k_m}(z_1 - z_1^0)^{k_1}(z_2 - z_2^0)^{k_2} \ldots (z_m - z_m^0)^{k_m},$$

which converges for sufficiently small values of $|z_i - z_i^0|$. It can be shown that

$$A_{k_1 k_2 \ldots k_m} = \frac{1}{k_1!\, k_2! \ldots k_m!}\left(\frac{\partial^{k_1 + k_2 + \cdots + k_m} F}{\partial z_1^{k_1} \partial z_2^{k_2} \ldots \partial z_m^{k_m}}\right)_{z_i = z_i^0}.$$

Let $\phi_i^{(k)}(x_1, x_2, \ldots, x_n)$ be the initial data which appear in the Cauchy problem for the system (1, 2) (cf. (2, 2)). We introduce the following abbreviated notation for the derivatives of these functions at some point $(x_1^0, x_2^0, \ldots, x_n^0)$:

$$\left(\frac{\partial^{k-k_0} \phi_i^{(k_0)}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}\right)_{x_i = x_i^0} = \phi^0_{i, k_0, k_1, k_2, \ldots, k_n}$$

$$(i = 1, 2, \ldots, N; \quad k_0 + k_1 + \ldots + k_n = k \leqslant n_i).$$

We now state the fundamental

CAUCHY-KOWALEWSKI THEOREM. *If all the functions* F_i *are analytic in some neighborhood of the point* $(t^0, x_1^0, x_2^0, \ldots, \phi^0_{j, k_0, k_1, \ldots, k_n}, \ldots)$, *and if all the functions $\phi_j^{(k)}$ are analytic in some neighborhood of the point $(x_1^0, x_2^0, \ldots, x_n^0)$, then the Cauchy problem has a unique analytic solution in some neighborhood of the point $(t^0, x_1^0, x_2^0, \ldots, x_n^0)$.*

3. We shall prove the Cauchy-Kowalewski theorem for arbitrary linear systems. The Cauchy problem for such systems is easily reducible to the Cauchy problem for linear first-order systems by means of a method whose use we shall illustrate for the case of a single second-order equation

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} a_{ij}(t, x_1, x_2, \ldots, x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} \\ + \sum_{i=1}^{n} a_{0i}(t, x_1, \ldots, x_n)\frac{\partial^2 u}{\partial t \partial x_i} \\ + \sum_{i=1}^{n} b_i(t, x_1, \ldots, x_n)\frac{\partial u}{\partial x_i} + b_0(t, x_1, \ldots, x_n)\frac{\partial u}{\partial t} \\ + c(t, x_1, \ldots, x_n)\, u + f(t, x_1, \ldots, x_n). \qquad (3, 2)\end{aligned}$$

Here $a_{ij} = a_{ji}$, b_i, c, f are analytic functions of their arguments in the neighborhood of the point $(t^0, x_1^0, \ldots, x_n^0)$.

The Cauchy problem for this equation consists in finding a solution satisfying the following initial conditions:

$$\left.\begin{aligned} u(t^0, x_1, x_2, \ldots, x_n) = \phi(x_1, \ldots, x_n), \\ u_t'(t^0, x_1, x_2, \ldots, x_n) = \phi_0(x_1, \ldots, x_n), \end{aligned}\right\} \qquad (4, 2)$$

where the functions ϕ and ϕ_0 are analytic in a neighborhood of the point $(x_1^0, x_2^0, \ldots, x_n^0)$. Without loss of generality we may assume that

$$t^0 = x_1^0 = \ldots = x_n^0 = 0,$$

for this can be achieved by means of a transformation of the independent variables which does not alter the form of the equation.

If the function $u(t, x_1, \ldots, x_n)$ satisfies equation (3, 2) and the initial conditions (4, 2), then, clearly, the functions

$$u, \quad u_0 = \partial u/\partial t, \quad u_k = \partial u/\partial x_k \quad (k = 1, 2, \ldots, n)$$

satisfy the equations

$$\frac{\partial u_0}{\partial t} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial u_i}{\partial x_j} + \sum_{i=1}^{n} a_{0i}\frac{\partial u_0}{\partial x_i} + \Sigma\, b_i u_i + cu + f, \qquad (5, 2)$$

$$\frac{\partial u_k}{\partial t} = \frac{\partial u_0}{\partial x_k} \quad (k = 1, 2, \ldots, n), \qquad (5, 2)'$$

$$\frac{\partial u}{\partial t} = u_0, \qquad (5, 2)''$$

and the initial conditions

$$u(0,x_1,\ldots,x_n)=\phi(x_1,\ldots,x_n), \tag{6, 2}$$

$$u_0(0,x_1,\ldots,x_n)=\phi_0(x_1,\ldots,x_n), \tag{6, 2$'$}$$

$$u_k(0,x_1,\ldots,x_n)=\phi_k(x_1,\ldots,x_n)=\partial\phi/\partial x_k \quad (k=1,\ldots,n) \tag{6, 2$''$}$$

We shall prove the converse of this theorem. If the functions $u, u_0, u_1, \ldots, u_n$ satisfy equations (5, 2) in some region G of the space $(t, x_1, \ldots, x_n)$ abutting on a region G_0 of the space $(x_1, x_2, \ldots, x_n)$ as well as the initial conditions (6, 2), (6, 2)$'$ and (6, 2)$''$, then $u(t, x_1, x_2, \ldots, x_n)$ satisfies equation (3, 2) and the initial conditions (4, 2) in all of the region G.

In fact, (5, 2)$''$ implies that in the whole region G

$$u_0=\partial u/\partial t.$$

Replacing u_0 in (5, 2)$'$ by $\partial u/\partial t$ we get

$$\frac{\partial u_k}{\partial t}=\frac{\partial^2 u}{\partial t\,\partial x_k} \quad \text{or} \quad \frac{\partial}{\partial t}\left[u_k-\frac{\partial u}{\partial x_k}\right]=0. \tag{7, 2}$$

Hence, the quantity $u_k-\partial u/\partial x_k$

is independent of t in all of the region G.

In view of the condition (6, 2)$''$,

$$u_k=\partial u/\partial x_k$$

in G_0. We may therefore conclude on the basis of (7, 2) that†

$$u_k=\partial u/\partial x_k \tag{8, 2}$$

in G. Putting $u_0=\partial u/\partial t$ and $u_k=\partial u/\partial x_k$ in (5, 2), we see that equation (3, 2) is satisfied in all of G.

We have thus shown that if

$$u_k=\partial u/\partial x_k$$

for $t=0$, then the system consisting of equations (5, 2), (5, 2)$'$ and (5, 2)$''$ or, briefly, the system (5, 2), is equivalent to equation (3, 2). Also, for arbitrary initial conditions the system (5, 2) has, in a sense, more solutions than equation (3, 2), for we can set up a one-to-one

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† Strictly speaking, the preceding considerations imply only that

$$u_k-\partial u/\partial x_k$$

does not depend on t on each segment of a line parallel to the t-axis if this segment is entirely contained in G. Consequently, $u_k-\partial u/\partial x_k=0$ in the part of G which can be covered with straight-line segments parallel to the t-axis, contained entirely in G and intersecting G_0. However, since the functions considered are analytic, it follows from a well-known theorem on analytic functions that $u_k-\partial u/\partial x_k=0$ in all of G.

correspondence between the solutions of (3, 2) and the solutions of the system (5, 2) with initial conditions satisfying the condition $u_k = \partial u/\partial x_k$.

4. We have seen that the Cauchy problem for the linear second-order equation (3, 2) is reducible to the Cauchy problem for the linear first-order system (5, 2). Using an entirely analogous procedure one can reduce an arbitrary system of the form (1, 2) to a system of first-order equations solved for the partial derivatives with respect to t of all unknown functions. Consequently, we will have proved the Cauchy-Kowalewski theorem for an arbitrary linear system which can be written in the form (1, 2) if we can prove it for an arbitrary linear system of the first order of the form

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{N} \sum_{k=1}^{n} a_{ij}^{(k)} \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{N} b_{ij} u_j + c_i \tag{9, 2}$$

with analytic coefficients and arbitrary analytic initial conditions

$$u_i(0, x_1, \ldots, x_n) = \phi_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, N). \tag{10, 2}$$

The case of arbitrary analytic functions ϕ_i is easily reduced to the case where all

$$\phi_i(x_1, \ldots, x_n) \equiv 0.$$

To achieve this we replace the unknown functions $u_i\ (t, x_1, \ldots, x_n)$ by new unknown functions

$$v_i(t, x_1, \ldots, x_n) = u_i(t, x_1, \ldots, x_n) - \phi_i(x_1, \ldots, x_n). \tag{11, 2}$$

The functions v_i will satisfy the system of equations

$$\frac{\partial v_i}{\partial t} = \sum_{j=1}^{N} \sum_{k=1}^{n} a_{ij}^{(k)} \frac{\partial v_j}{\partial x_k} + \sum_{j=1}^{N} b_{ij} v_j + \left(c_i + \sum_{j=1}^{N} \sum_{k=1}^{n} a_{ij}^{(k)} \frac{\partial \phi_j}{\partial x_k} + \sum_{j=1}^{N} b_{ij} \phi_j \right), \tag{12, 2}$$

which is entirely analogous to the system (9, 2), and the initial conditions

$$v_i(0, x_1, x_2, \ldots, x_n) \equiv 0. \tag{13, 2}$$

It is clear that a proof of the existence of a solution of the Cauchy problem for the system (12, 2) with initial conditions which are identically zero would imply the existence of a solution of our original problem.

To simplify notation we shall assume that the original functions $u_i(t, x_1, \ldots, x_n)$ satisfy the initial conditions

$$u_i(0, x_1, \ldots, x_n) \equiv 0. \tag{14, 2}$$

5. We first prove the uniqueness in some neighborhood of the point O $(t = 0, x_1 = 0, \ldots, x_n = 0)$ of the analytic solution of the Cauchy

problem for the system (9, 2) and the initial conditions (14, 2). In other words, we wish to show that for no neighborhood of the point O do there exist two different analytic solutions of the system (9, 2) satisfying, for $t=0$, the same initial conditions (14, 2). Since the functions $u_i(t, x_1, \ldots, x_n)$ are assumed to be analytic in some neighborhood of the origin, they can be expanded about the origin in a power series in $t, x_1, \ldots, x_n$. In this expansion the coefficient $a^i_{k_0 k_1 \ldots k_n}$ of $t^{k_0} x_1^{k_1} \ldots x_n^{k_n}$ is equal to

$$\frac{1}{k_0!\, k_1! \ldots k_n!} \left(\frac{\partial^{k_0+k_1+\cdots+k_n} u_i}{\partial t^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}}\right)_{t=x_1=\ldots=x_n=0}.$$

We shall have proved the uniqueness of the solution of the Cauchy problem if we can prove that the initial conditions (14, 2) determine uniquely the coefficients of the power-series expansions of the functions u_i satisfying the system (9, 2), or, which is the same, if we can prove that these conditions determine uniquely the values of the derivatives of the u_i at O $(t=0,\ x_1=0, \ldots, x_n=0)$. We shall determine these derivatives one by one. The initial conditions determine uniquely the values at O of all derivatives of the form

$$\left(\frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}\right)_{t=x_1=\ldots=x_n=0}. \tag{15, 2}$$

All these derivatives are equal to zero, for the identities (14, 2) may be differentiated with respect to $x_1, x_2, \ldots, x_n$. Let us assume the existence of a solution of the Cauchy problem and let us replace the u_j in (9, 2) by the alleged solution functions. Differentiating the resulting identities k_1 times with respect to x_1, k_2 times with respect to $x_2, \ldots, k_n$ times with respect to x_n, we obtain on the left sides derivatives of the form

$$\frac{\partial^{k+1} u_i}{\partial t\, \partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \tag{16, 2}$$

and on the right sides, derivatives with respect to $x_1, x_2, \ldots, x_n$ of the unknown functions and of the coefficients of the equation, i.e. quantities which are uniquely determined at the point O by the equations and by the initial conditions. The identities so obtained determine the values at the point O of the derivatives of the form (16, 2) (i.e. derivatives involving one differentiation with respect to t).

We now differentiate each of the identities (9, 2) once with respect to t, k_1 times with respect to $x_1, \ldots, k_n$ times with respect to x_n. Then, on the right sides we obtain expressions consisting of derivatives of the u_i of the form (16, 2) and (15, 2) and of derivatives of the coefficients a_{ij}, b_{ij}, c_i. On the left sides we obtain derivatives of the form

$$\frac{\partial^{k+2} u_i}{\partial t^2 \partial x_1^{k_1} \dots \partial x_n^{k_n}} \tag{17, 2}$$

(i.e. derivatives involving two differentiations with respect to t). Since we have already shown that the values at the point O of all derivatives of the form (16, 2) and (15, 2) are uniquely determined by equations (9, 2) and by the initial conditions (14, 2), it follows that the values at the point O of all derivatives of the form (17, 2) are also uniquely determined. Continuing this process we shall find that the values at the point O of all derivatives of the u_i are uniquely determined by equations (9, 2) and the initial conditions (14, 2). Now, the values at the fixed point O of all derivatives of an analytic function $u_i(t, x_1, \dots, x_n)$ determine uniquely the values of the coefficients of its power-series expansion about the point O and so determine completely the values of the function itself in some neighborhood of the point O. Thus, two analytic solutions of the system (9, 2) satisfying the same initial conditions (14, 2) necessarily coincide in some neighborhood of the origin. This proves the uniqueness of the solution of the Cauchy problem for the system (9, 2) in the class of analytic functions.

6. We showed in para. 5 that the initial conditions determine completely the coefficients of the expansion of the functions u_i in power series in $t, x_1, \dots, x_n$. To prove the existence of a solution of the Cauchy problem it suffices to prove that the power series with coefficients defined in para. 5 converge in some neighborhood of the point O. Indeed, if these series were known to converge, they would represent analytic functions $u_i(t, x_1, \dots, x_n)$ which would vanish at O together with all their partial derivatives with respect to $x_1, x_2, \dots, x_n$ (see (15, 2) and the sentence that follows it). Consequently, for $t = 0$, they would be identically zero in $x_1, x_2, \dots, x_n$ and so would satisfy the initial conditions (14, 2). That these functions would also satisfy the system (9, 2) follows from the very method of their construction. This method of construction implies that, at O, the values of the left sides of equations (9, 2) and of their derivatives with respect to $t, x_1, \dots, x_n$, with u_i replaced by our functions, coincide, respectively, with the values of the right sides of these equations and of their appropriate derivatives. Hence, in some neighborhood of the origin, the left sides of our equations are identically equal to the right sides.

To prove the convergence of the power series for the functions u_i we use the *method of majorants.*

7. Let the function $\phi(t, x_1, \ldots, x_n)$ be analytic in some neighborhood of the origin. Let $\psi(t, x_1, \ldots, x_n)$ be a function analytic in the same neighborhood and such that the coefficients of its expansion in powers of $t, x_1, \ldots, x_n$ are non-negative and not smaller than the absolute values of the corresponding coefficients of the power-series expansion of $\phi(t, x_1, \ldots, x_n)$. Then the function ψ is said to *majorize* the function ϕ.

LEMMA. *For each function $\phi(t, x_1, \ldots, x_n)$ analytic in some neighborhood of the origin there exists a function which majorizes it.*

Proof. Let

$$\phi(t, x_1, \ldots, x_n) = \Sigma c_{k_0 k_1 \ldots k_n} t^{k_0} x_1^{k_1} \ldots x_n^{k_n}. \tag{18, 2}$$

The series on the right converges at some point

$$t = a_0,\ x_1 = a_1, \ldots, x_n = a_n \quad (|a_i| > 0).$$

There exists a number M such that

$$|c_{k_0 k_1 \ldots k_n} a_0^{k_0} a_1^{k_1} \ldots a_n^{k_n}| \leqslant M$$

for any set of non-negative integers $k_0, k_1, \ldots, k_n$. Consequently, for all $k_0, k_1, \ldots, k_n$

$$|c_{k_0 k_1 \ldots k_n}| \leqslant \frac{M}{|a_0|^{k_0} |a_1|^{k_1} \ldots |a_n|^{k_n}}.$$

Hence the function

$$S = \frac{M}{\left(1 - \frac{t}{|a_0|}\right)\left(1 - \frac{x_1}{|a_1|}\right) \cdots \left(1 - \frac{x_n}{|a_n|}\right)}$$

$$= M\left[\sum_{k_0=0}^{\infty} \left(\frac{t}{|a_0|}\right)^{k_0} \sum_{k_1=0}^{\infty} \left(\frac{x_1}{|a_1|}\right)^{k_1} \cdots \sum_{k_n=0}^{\infty} \left(\frac{x_n}{|a_n|}\right)^{k_n}\right]$$

$$= \sum_{k_0, k_1, \ldots, k_n} \frac{M}{|a_0|^{k_0} |a_1|^{k_1} \ldots |a_n|^{k_n}} t^{k_0} x_1^{k_1} \ldots x_n^{k_n} \tag{19, 2}$$

majorizes the function $\phi(t, x_1, \ldots, x_n)$.

It is possible to give another method for constructing a majorizing series. Thus, for instance, the function $\phi(t, x_1, \ldots, x_n)$, represented by the series (18, 2), is majorized by the function

$$\frac{M}{1 - \frac{t + x_1 + \ldots + x_n}{a}}, \quad \text{where} \quad a = \min(|a_0|, |a_1|, \ldots, |a_n|)$$

and $(a_0, a_1, \ldots, a_n)$ denotes a point at which the series (18, 2) converges.

In fact, if $|t + x_1 + \ldots + x_n| < a$, this function has the series representation

$$M \sum_{k=0}^{\infty} \left(\frac{t + x_1 + \ldots + x_n}{a}\right)^k$$
$$= M \sum_{k=0}^{\infty} \frac{1}{a^k} \sum_{k_0 + k_1 + \ldots + k_n = k} \frac{k!}{k_0! \, k_1! \ldots k_n!} t^{k_0} x_1^{k_1} \ldots x_n^{k_n}. \quad (20, 2)$$

Since

$$\frac{(k_0 + k_1 + \ldots + k_n)!}{k_0! \, k_1! \ldots k_n!} \geqslant 1, \quad \frac{1}{a^k} \geqslant \frac{1}{|a_0|^{k_0} |a_1|^{k_1} \ldots |a_n|^{k_n}},$$

it follows that the coefficients of our series are positive and not smaller than the corresponding coefficients of the series given by (19, 2). Consequently, the function (20, 2) also majorizes (18, 2).

Another majorant of the function $\phi(t, x_1, \ldots, x_n)$ is the function

$$\frac{M}{1 - \dfrac{(t/\alpha) + x_1 + \ldots + x_n}{a}} = M \sum_{k=0}^{\infty} \frac{[(t/\alpha) + x_1 + \ldots + x_n]^k}{a^k}, \quad (21, 2)$$

where a has the same value as before and $0 < \alpha < 1$.

If we expand $[(t/\alpha) + x_1 + \ldots + x_n]^k$ in powers of $t, x_1, \ldots, x_n$, we obtain a series with positive coefficients which are greater than the corresponding coefficients of the expansion in powers of t, x_1, ..., x_n of the function (20, 2). This is so because the coefficients of the former series are obtained from the coefficients of the latter series by multiplying by $(1/\alpha)^{k_0}$, where $0 < \alpha < 1$.

8. We now turn to the proof of the existence of a solution of the Cauchy problem for the system (9, 2) with initial conditions (14, 2). We shall refer to this problem as 'problem (I)' and to the system (9, 2) as 'system I'.

Let us assume that we have somehow majorized the coefficients of the system and the initial Cauchy data. We shall thus obtain a new system and a new Cauchy problem (to which we shall refer as 'system II' and 'problem (II)' respectively). We shall show that the analytic solution of 'problem (II)' majorizes the analytic solution of 'problem (I)'. To this end we must show that if the solution of 'problem (I)' has a series representation about the origin given by

$$u_i = \Sigma a^{(i)}_{k_0 k_1 \ldots k_n} t^{k_0} x_1^{k_1} \ldots x_n^{k_n}, \quad (22, 2)$$

and the solution of 'problem (II)' has a series representation given by

$$U_i = \Sigma A^{(i)}_{k_0 k_1 \dots k_n} t^{k_0} x_1^{k_1} \dots x_n^{k_n}, \tag{23, 2}$$

then

$$|a^{(i)}_{k_0 k_1 \dots k_n}| \leqslant A^{(i)}_{k_0 k_1 \dots k_n}. \tag{24, 2}$$

For the case $k_0 = 0$, these inequalities are a direct consequence of the fact that the initial data of 'problem (II)' majorize the initial data of 'problem (I)'. For $k_0 > 0$, the coefficients $a^{(i)}_{k_0 k_1 \dots k_n}$ are obtained from the $a^{(i)}$'s with smaller index k_0 and from the values at O of the coefficients of 'system I' and their derivatives by means of addition and multiplication. An entirely analogous statement holds for the coefficients $A^{(i)}_{k_0 k_1 \dots k_n}$ with '$a^{(i)}$' replaced by '$A^{(i)}$' and with 'system I' replaced by 'system II'. It is now easy to check that if the inequalities (24, 2) hold for $k_0 < k$, then they also hold for $k_0 = k$. This implies their correctness for all the coefficients which appear in the expansions (22, 2) and (23, 2).

Thus, the existence of a solution of 'problem (II)' (i.e. the convergence of the series (23, 2)) implies the existence of a solution of 'problem (I)' (i.e. the convergence of the series (22, 2)). As for 'problem (II)' it may be formulated with a great degree of arbitrariness in the sense that we are not restricted in the choice of majorants of the coefficients and of the initial data of 'problem (I)'. We shall try to formulate 'problem (II)' in a manner which will permit us to solve it with little difficulty. With this end in view we choose the numbers M and a so that the function

$$\frac{M}{1 - \dfrac{(t/\alpha) + x_1 + \dots + x_n}{a}}$$

majorizes all the coefficients of the system with the exception of the free (i.e. not involving derivatives) terms which are to be majorized by a function of the form†

$$\frac{M_1}{1 - \dfrac{(t/\alpha) + x_1 + x_2 + \dots + x_n}{a}}.$$

This can be done, for there exist majorizing functions of the above form for each coefficient and, in order to construct a general majorizing function, it suffices to give M and M_1 the largest of all values which are assigned in the case of the individual coefficients and to a—the smallest of all values which are assigned in the case of the individual coefficients. With this choice of the values of the numbers M, M_1, and a, we can set down the majorizing system as follows:

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† The possibility of choosing M_1 independently of M will turn out to be very useful in the sequel (cf. the remark at the end of this paragraph).

$$\frac{\partial U_i}{\partial t}=\frac{M}{1-\dfrac{(t/\alpha)+x_1+\ldots+x_n}{a}}\left[\sum_{j,k}\frac{\partial U_j}{\partial x_k}+\sum_j U_j+m\right]. \tag{25, 2}$$

Here $0<\alpha<1$ is still to be chosen and $m=M_1/M$.

Without as yet fixing initial data, we shall seek a solution of our system of the form

$$\begin{aligned}U_1(t,x_1,\ldots,x_n)\equiv U_2(t,x_1,\ldots,x_n)\equiv\ldots\equiv U_N(t,x_1,\ldots,x_n)\\ =U(t,x_1,\ldots,x_n)=U[(t/\alpha)+x_1+\ldots+x_n]=U(z),\\ z=(t/\alpha)+x_1+\ldots+x_n.\end{aligned}$$

Putting the assumed solution $U(z)$ in (25, 2), we see that it must necessarily satisfy the relation

$$\frac{1}{\alpha}\frac{dU}{dz}=A(z)\left(Nn\frac{dU}{dz}+NU+m\right), \tag{26, 2}$$

with

$$A(z)=\frac{M}{1-(z/a)}.$$

This equation belongs to the class of equations with separable variables and can be written in the form

$$\frac{dU}{\dfrac{N}{m}U+1}=\frac{mA(z)\,dz}{\dfrac{1}{\alpha}-NnA(z)}=B(z)\,dz.$$

We now choose the positive number α so small that

$$(1/\alpha)-Nn\,|A(z)|>0 \tag{27, 2}$$

in some neighborhood of the point $z=0$. Then $B(z)$ will be an analytic function in that neighborhood.

We shall show that the special solution

$$U(z)=\frac{\exp\left\{(N/m)\displaystyle\int_0^z B(\xi)\,d\xi\right\}-1}{N}\cdot m$$

of (26, 2) is the required majorant for the solution of 'problem (1)'.

Since the functions

$$U_i(t,x_1,\ldots,x_n)=U[(t/\alpha)+x_1+\ldots+x_n]$$

satisfy the system (25, 2) which, in turn, majorizes the original system, we shall have proved our contention if we can show that for $t=0$ we can write $U(z)$ as a power series in $x_1, x_2, \ldots, x_n$ with positive coefficients, or, what amounts to the same thing, that $U(z)$ majorizes zero (i.e. the initial data of 'problem (I)').

In fact

$$A(z)=\frac{M}{1-(z/a)}$$

is a function whose expansion in powers of z has non-negative coefficients. Consequently, the expansion in powers of z of the function

$$B(z)=\frac{m\alpha A(z)}{1-\alpha A(z)\,Nn}$$
$$=m\alpha A(z)\left[1+\alpha NnA(z)+\alpha^2N^2n^2A^2(z)+\ldots\right]$$

has non-negative coefficients. It follows that

$$C(z)=\frac{N}{m}\int_0^z B(z)\,dz, \quad \exp\{C(z)\}-1=C(z)+\frac{C^2(z)}{2!}+\ldots, \quad U(z)$$

also possess this property. Therefore, the coefficients of the expansion of $U(x_1+x_2+\ldots+x_n)$ in powers of $x_1, x_2, \ldots, x_n$ are non-negative, i.e. $U(0, x_1, x_2, \ldots, x_n)$ is actually a majorant of zero. Hence the functions

$$U_i(t, x_1, \ldots, x_n)=U[(t/\alpha)+x_1+\ldots+x_n]$$

are a solution of a certain 'problem (II)'. The analyticity of this solution is a consequence of the fact that $U(z)$, as shown above, can be represented as a power series in z and, hence, as a power series in $t, x_1, \ldots, x_n$. This implies, as shown above, the convergence of the power series (22, 2) which represent the solution of the original problem.

This concludes the proof of the Cauchy-Kowalewski theorem for linear systems.

Remark. It is clear from the proof of the theorem that the series which represent the solution of the Cauchy problem for the system (9, 2) converge at least in the region of convergence of the series giving the solution of the majorizing problem. It follows that a solution of the Cauchy problem for the system (9, 2) and the initial functions ϕ (which are not necessarily equal to zero) exists at least in some region

$$|t/\alpha|<\rho, \ |x_1|<\rho, \ldots, |x_n|<\rho, \quad (\rho>0),$$

provided that the coefficients of the system (9, 2) as well as the initial functions are analytic in some region

$$|t| \leqslant R, \quad |x_i| \leqslant R \quad (i = 1, 2, \ldots, n;\ R > 0).$$

Here ρ and α depend only on R and on the number M for which we may choose an upper bound of the moduli of all the coefficients of the system (9, 2) with the exception of c_i. M does not in any way depend on the values of the initial functions ϕ_i and on the free terms of the equation, for neither α nor the region in the z plane in which (27, 2) holds depends on these values.

§ 3. The generalized Cauchy problem. Characteristics.

1. *The generalized Cauchy problem.* Let there be given a system of N equations with N unknown functions $u_1, u_2, \ldots, u_N$

$$\Phi_i\left(x_0, x_1, \ldots, x_n;\ u_1, \ldots, u_N, \ldots, \frac{\partial^k u_j}{\partial x_0^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \ldots\right) = 0$$

$$(i, j = 1, 2, \ldots, N). \quad (1, 3)$$

For each function u_i there exists a highest order n_i of the partial derivatives of u_i with respect to the independent variables $x_0, x_1, \ldots, x_n$ which appear in the system (1, 3). In the region of points $(x_0, x_1, \ldots, x_n)$ under consideration there is given a sufficiently smooth n-dimensional surface S, and through each point of S there passes a curve l not tangent to S which is assumed to vary sufficiently smoothly as we go from point to point on S. The values of the functions u_i and of their derivatives of orders up to $n_i - 1$ in the direction of the curves l are prescribed on the surface S. These conditions, as prescribed on S, are a generalization of the Cauchy conditions (initial data) considered in the previous section. We are required to find in some neighborhood of the surface S a solution $u_1, u_2, \ldots, u_N$ of the system (1, 3) satisfying the conditions prescribed on S.

2. We shall now try to reduce this problem to the Cauchy problem formulated in the preceding section. For the sake of definiteness and simplicity we shall at first limit ourselves to the consideration of the linear system

$$\sum_{j, k_0, k_1, \ldots, k_n} A_{ij}^{(k_0 \ldots k_n)}(x_0, x_1, \ldots, x_n) \frac{\partial^{n_j} u_j}{\partial x_0^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}}$$

$$+ \ldots + f_i(x_0, x_1, \ldots, x_n) = 0$$

$$(i, j = 1, 2, \ldots, N). \quad (2, 3)$$

Here we have written out only terms with highest order derivatives of the unknown functions. We now construct a set of $n+1$ sufficiently smooth surfaces

$$\psi_k(x_0, x_1, \ldots, x_n) = \xi_k \quad (k = 0, 1, \ldots, n) \tag{3, 3}$$

(ξ_k are constant parameters) such that

$$\xi_0 = 0 \tag{4, 3}$$

is the equation of the surface S; for some small ϵ, the curves

$$-\epsilon \leqslant \xi_0 \leqslant \epsilon; \quad \xi_1 = c_1, \ldots, \xi_n = c_n \tag{5, 3}$$

coincide with the curves l; and the Jacobian

$$\left| \frac{\partial \psi_k}{\partial x_i} \right|$$

is different from zero in some neighborhood of the surface S. Then, in some neighborhood of the surface S, it is possible to replace the coordinates $x_0, x_1, \ldots, x_n$ by new coordinates $\xi_0, \xi_1, \ldots, \xi_n$. In the new coordinates, (4, 3) is the equation of the surface S, and (5, 3) are the equations of the curves l.

The construction of a set of surfaces with the above properties can be realized in the following manner. Let the equation of the surface S be of the form

$$F(x_0, x_1, \ldots, x_n) = 0.$$

On this surface we choose a smooth system of coordinates $\xi_1, \xi_2, \ldots, \xi_n$. The set of curves l, one of which is to go through each point of S, will depend on n parameters for which we can take the coordinates $\xi_1, \xi_2, \ldots, \xi_n$ of a point on the surface. Let ξ_0 denote the coordinate which varies along a curve l. Then the parametric equations of this curve are of the form

$$x_0 = X_0(\xi_0, \xi_1, \ldots, \xi_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = X_n(\xi_0, \xi_1, \ldots, \xi_n).$$

Here ξ_0 is the parameter of a point on the curve l, and ξ_1, ξ_2, …, ξ_n are the parameters of the curve itself.

If the curves l represent a regular family of curves in some neighborhood of the surface S (i.e. if through each point of this neighborhood there goes one and only one curve l), then the numbers $\xi_0, \xi_1, \ldots, \xi_n$ may be taken as the new coordinates of the point $(x_0, x_1, \ldots, x_n)$.

We now apply our change of coordinates to equations (2, 3). We shall be primarily interested in those terms of the transformed equations which contain derivatives of order n_i (the highest order) of the u_i with respect to ξ_0. Writing out only those terms we obtain

$$\frac{\partial^{n_i} u_i}{\partial x_0^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}} = \frac{\partial^{n_i} u_i}{\partial \xi_0^{n_i}} \left(\frac{\partial \xi_0}{\partial x_0}\right)^{k_0} \left(\frac{\partial \xi_0}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial \xi_0}{\partial x_n}\right)^{k_n} \cdots + .$$

Therefore, writing out only the highest order derivatives of the functions u_i with respect to ξ_0 in the transformed equations (2, 3) we get

$$\sum_{\substack{j=1 \\ k_0+\ldots+k_n=n_j}}^{N} A_{ij}^{(k_0 \ldots k_n)} \left(\frac{\partial \xi_0}{\partial x_0}\right)^{k_0} \cdots \left(\frac{\partial \xi_0}{\partial x_n}\right)^{k_n} \frac{\partial^{n_j} u_j}{\partial \xi_0^{n_j}} + \ldots = 0 \quad (i = 1, 2, \ldots, N). \tag{6, 3}$$

In order that it be possible to solve these equations for $\partial^{n_j} u_j / \partial \xi_0^{n_j}$ in the neighborhood of the surface S, regardless of the form of the terms of these equations which have not been written out, it is necessary and sufficient that the determinant

$$\left| \sum_{k_0+\ldots+k_n=n_j} A_{ij}^{(k_0 k_1 \ldots k_n)} \left(\frac{\partial \xi_0}{\partial x_0}\right)^{k_0} \cdots \left(\frac{\partial \xi_0}{\partial x_n}\right)^{k_n} \right| \quad (i, j = 1, 2, \ldots, N)$$

be different from zero at all points of S. Then, in view of the continuity of the coefficients $A_{ij}^{(k_0 \ldots k_n)}$ and of the derivatives $\partial \xi_0 / \partial x_k$, this determinant will be different from zero in some neighborhood of the surface S in the space $(x_0, x_1, \ldots, x_n)$.

The equation

$$\left| \sum_{k_0+\ldots+k_n=n_j} A_{ij}^{(k_0 k_1 \ldots k_n)} (x_0, \ldots, x_n) \alpha_0^{k_0} \alpha_1^{k_1} \ldots \alpha_n^{k_n} \right| = 0$$
$$(i, j = 1, 2, \ldots, N) \tag{7, 3}$$

is called the *characteristic equation* of the system (2, 3); here $\alpha_0, \alpha_1, \ldots, \alpha_n$ are certain parameters. The plane

$$\sum_{k=0}^{n} \alpha_k (x_k - x_k^0) = 0$$

is said to be *characteristic* at the point $(x_0^0, \ldots, x_n^0)$ for the system (2, 3) provided that†

$$\left| \sum_{k_0+\ldots+k_n=n_j} A_{ij}^{(k_0 \ldots k_n)} (x_0^0, x_1^0, \ldots, x_n^0) \alpha_0^{k_0} \ldots \alpha_n^{k_n} \right| = 0.$$

---

† Since equation (7, 3) is homogenous with respect to the quantities $\alpha_0, \alpha_1, \ldots, \alpha_n$, the latter can be normalized by assuming that $\sum_{k=0}^{n} \alpha_k^2 = 1$, say. Then α_k will simply be the cosine of the angle between the normal to the characteristic plane and the x_k-axis.

The surface $\phi(x_0, x_1, \ldots, x_n) = 0$

is called a *characteristic surface*, or, simply, a *characteristic* of the system (2, 3) if at each point of this surface

$$\left| \sum_{k_0+\ldots+k_n=n_j} A_{ij}^{(k_0 k_1 \ldots k_n)}(x_0, x_1, \ldots, x_n) \times (\partial\phi/\partial x_0)^{k_0} (\partial\phi/\partial x_1)^{k_1} \ldots (\partial\phi/\partial x_n)^{k_n} \right| = 0.$$

The above definitions imply that a surface is characteristic at a point P if and only if its tangent plane at the point P is characteristic at P.

3. Let us suppose that the surface S, which appears in the formulation of the generalized Cauchy problem, is nowhere characteristic for the system (2, 3). It is by now clear that after replacing the coordinates $x_0, x_1, \ldots, x_n$ by the coordinates $\xi_0, \xi_1, \ldots, \xi_n$ (see para. 2), it is always possible to solve the transformed system (6, 3) for the highest order derivatives of the u_i with respect to ξ_0. We then obtain the following system of equations:

$$\frac{\partial^{n_i} u_i}{\partial \xi_0^{n_i}} = \sum_{j,k} B_{ij}^{(k_0 \ldots k_n)}(\xi_0, \ldots, \xi_n) \frac{\partial^k u_j}{\partial \xi_0^{k_0} \ldots \partial \xi_n^{k_n}} + F_i(\xi_0, \ldots, \xi_n) \quad (8, 3)$$

$$(i, j = 1, 2, \ldots, N; \quad k = k_0 + k_1 + \ldots + k_n \leqslant n_j; \quad k_0 < n_j).$$

At the same time the conditions prescribed on the surface S become:

$$\left(\frac{\partial^k u_i}{\partial \xi_0^k}\right)_{\xi_0=0} = \phi_i^{(k)}(\xi_1, \ldots, \xi_n) \quad (k = 0, 1, \ldots, n_i - 1; \quad i = 1, 2, \ldots, N).$$

We may, therefore, conclude that if the surface S is nowhere characteristic, then the generalized Cauchy problem can be reduced to the Cauchy problem. The transition from the first of these problems to the second is fully reversible; to each sufficiently smooth† solution of one problem there corresponds a unique smooth† solution of the other problem.

We recall that in the preceding section we discussed the problem of solving a system with analytic coefficients and analytic initial conditions. For the system (8, 3) and for the Cauchy problem associated with this system to satisfy these requirements, it is sufficient that the following conditions hold:

(a) The coefficients of the system (2, 3) are analytic functions of $x_0, \ldots, x_n$.

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† We need only require the continuity of the derivatives of the u_i of orders up to and including n_i.

(*b*) The functions $\xi_k = \psi_k(x_0, x_1, \ldots, x_n)$ are analytic functions of their arguments.

The possibility of choosing analytic ψ_k is connected with the character of the surface S and of the family of curves l. A surface S and a family of curves l for which such choice is possible are called, respectively, an analytic surface and an analytic family of curves.

If the surface is given by means of an equation $F(x_0, x_1, \ldots, x_n) = 0$, then it is analytic if $F(x_0, x_1, \ldots, x_n)$ is an analytic function of its arguments and if the surface has no singular points (i.e. points at which all first derivatives of F vanish). The family of normals to an analytic surface is an analytic family.

(*c*) The initial data are analytic functions of $\xi_1, \ldots, \xi_n$.

It follows from the Cauchy-Kowalewski theorem that if conditions *a*, *b* and *c* are satisfied then the generalized Cauchy problem always has a unique solution in some neighborhood of the surface S provided S is nowhere characteristic.

If, on the other hand, the surface S is characteristic at some point A, i.e. if at the point A of the surface

$$\xi_0(x_0, \ldots, x_n) = 0,$$

the equality

$$\left| \sum_{k_0 + \ldots + k_n = n_j} A_{ij}^{(k_0 \ldots k_n)} \left(\frac{\partial \xi_0}{\partial x_0}\right)^{k_0} \cdots \left(\frac{\partial \xi_0}{\partial x_n}\right)^{k_n} \right| = 0. \tag{9, 3}$$

holds, then, generally speaking, we cannot prescribe the values of the u_i and of their derivatives arbitrarily and at the same time expect the generalized Cauchy problem to have a solution. To see this let us rearrange equations (6, 3) by putting all terms containing derivatives of order n_i of the functions u_i with respect to ξ_0 on the left sides and the remaining terms on the right sides of these equations. It then follows from (9, 3) that the left sides of our equations are linearly dependent at A. This implies that the right sides of our equations, which are fully determined by the values of the u_i and of their derivatives prescribed on S, must likewise be linearly dependent at A. This, in turn, relates the initial data in a definite manner provided the relation of linear dependence connecting the right sides of our equations is not identically satisfied for all values of the functions u_i and of their derivatives on S. In the latter case, and also in the case when the Cauchy conditions are prescribed on the characteristic surface S in such a manner that the system can be solved for the highest order derivatives with respect to ξ_0 regarded as independent variables, such solution is not unique at the point A.

We shall now give a number of examples for the determination of characteristic surfaces for equations and for systems of equations. When analyzing these examples we shall always assume that

$$\Sigma\alpha_i^2 = 1, \tag{10,3}$$

i.e. that the α_i always denote the cosine of the angle between the x_i-axis and the normal to a characteristic hyperplane.

Example 1. For the Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0,$$

the relation (7, 3) assumes the form

$$\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = 0.$$

This relation together with (10, 3) imply that the Laplace equation has no real characteristics.

Example 2. For the wave equation

$$\frac{\partial^2 u}{\partial x_0^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

the relation (7, 3) takes the form

$$\alpha_0^2 - \alpha_1^2 - \alpha_2^2 = 0.$$

Since, according to (10, 3), we must also have

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 1,$$

it follows that $2\alpha_0^2 = 1$, $\alpha_0 = \pm 2^{-\frac{1}{2}}$. This means that the tangent planes to all characteristic surfaces form an angle of 45° with the x_0-axis. Using this property of the characteristic surfaces, one can easily picture the form of the characteristic surfaces going through definite curves in the plane $x_0 = \text{const}$. Thus, for example, if l is a straight line in that plane, the characteristic surface going through l is the plane through l inclined at 45° with respect to the plane $x_0 = \text{const}$. Again, if K is the circumference of a circle in the plane $x_0 = \text{const.}$, the characteristic surfaces through K will be circular cones whose axes are parallel to the x_0-axis and whose generators form a 45° angle with the plane $x_0 = \text{const.}$ (or, which is the same thing, with the x_0-axis).

It is easily seen that entirely analogous results hold for the so-called general wave equation in n-dimensional space

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\dots+\frac{\partial^2 u}{\partial x_n^2}.$$

Example 3. For the heat equation

$$\frac{\partial u}{\partial x_0}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\dots+\frac{\partial^2 u}{\partial x_n^2},$$

the relation (7, 3) takes the form

$$\alpha_1^2+\dots+\alpha_n^2=0.$$

Using (10, 3) we conclude that $\alpha_0^2=1$. Therefore, the characteristic surfaces are the hyperplanes $x_0=\text{const.}$

Example 4. For the equation

$$a_1(x_1,\dots,x_n)\frac{\partial u}{\partial x_1}+a_2(x_1,\dots,x_n)\frac{\partial u}{\partial x_2}+\dots+a_n(x_1,\dots,x_n)\frac{\partial u}{\partial x_n}=0,$$

(7, 3) takes the form

$$a_1(x_1,\dots,x_n)\,\alpha_1+a_2(x_1,\dots,x_n)\,\alpha_2+\dots+a_n(x_1,\dots,x_n)\,\alpha_n=0.$$

Hence all hyperplanes going through the point $(x_1,\dots,x_n)$ and the vector $(a_1,\dots,a_n)$ emanating from $(x_1,\dots,x_n)$ are characteristic at $(x_1,\dots,x_n)$.

Example 5. For the system of equations in two unknowns

$$\sum_{j=1}^{n} a_{ij}(x_1,x_2)\,(\partial u_j/\partial x_1)+\sum_{j=1}^{n} b_{ij}(x_1,x_2)\,(\partial u_j/\partial x_2)+\sum_{j=1}^{n} c_{ij}(x_1,x_2)\,u_j=0,$$
$$(i=1,2,\dots,n),$$

the relation (7, 3) assumes the form

$$|\,\alpha_1 a_{ij}(x_1,x_2)+\alpha_2 b_{ij}(x_1,x_2)\,|=0.$$

In this case, the characteristic curves are the curves along which dx_2/dx_1, i.e. $-(\partial\phi/\partial x_1):(\partial\phi/\partial x_2)$, is equal to a root k of the equation

$$|-ka_{ij}(x_1,x_2)+b_{ij}(x_1,x_2)\,|=0.$$

Here we are assuming that $\phi(x_1,x_2)=0$ is the equation of the characteristic curve.

4. In the case of non-linear systems of the form

$$\Phi_i\left(x_0,\dots,x_n;\,u_1,\dots,u_N,\dots,\frac{\partial^k u_j}{\partial x_0^{k_0}\partial x_1^{k_1}\dots\partial x_n^{k_n}},\dots\right)=0 \quad (11,3)$$
$$(i,j=1,\dots,N;\quad k=k_0+k_1+\dots+k_n\leqslant n_j)$$

the *characteristic equation* is the equation

$$\left| \sum_{k_0+k_1+\ldots+k_n=n_j} \frac{\partial \Phi_i}{\partial \left\{ \dfrac{\partial^{n_j} u_j}{\partial x_0^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}} \right\}} \alpha_0^{k_0} \ldots \alpha_n^{k_n} \right| = 0. \qquad (12,3)$$

A surface
$$\phi(x_0, \ldots, x_n) = 0 \qquad (13,3)$$

is called *characteristic* for the system (11, 3) and *for a given solution* $u_1, u_2, \ldots, u_N$ if the solution functions $u_1, u_2, \ldots, u_N$ satisfy the equation

$$\left| \sum_{k_0+k_1+\ldots+k_n=n_j} \frac{\partial \Phi_i}{\partial \left\{ \dfrac{\partial^{n_j} u_j}{\partial x_0^{k_0} \ldots \partial x_n^{k_n}} \right\}} \left(\frac{\partial \phi}{\partial x_0}\right)^{k_0} \ldots \left(\frac{\partial \phi}{\partial x_n}\right)^{k_n} \right| = 0$$

at all points of the surface.

The definition of *characteristic plane* for the system (11, 3) at a given point of the space $(x_0, \ldots, x_n)$ and for a given solution $u_1, u_2, \ldots, u_N$ is given in a manner entirely analogous to that in which this concept has been defined in the less general case. In the case of non-linear systems it makes sense to speak of the characteristicity of the plane
$$\Sigma \alpha_k (x_k - x_k^0) = 0$$
at a given point only for a *definite solution* $u_1, u_2, \ldots, u_N$ of the system (11, 3) for, in this case, the coefficients of the equation (12, 3) depend, in general, on the functions u_i and on their derivatives of order up to n_i.

Using the method of proof employed in para. 3 we can prove an interesting fact. As in para. 3, let the Cauchy conditions be prescribed on some analytic surface. Let us also assume that all functions considered there are analytic. Since we are not assuming (11, 3) to be linear, the system of equations for $\partial^{n_i} u_i / \partial \xi_0^{n_i}$ obtained by going over to the coordinates $\xi_0, \xi_1, \ldots, \xi_n$ (cf. para. 3) will be non-linear. Let this system of equations be denoted by Σ. This system has, in general, more than one solution for

$$\partial^{n_i} u_i / \partial \xi_0^{n_i} \quad (i = 1, 2, \ldots, N)$$

considered as functions of the independent variables

$$\xi_0, \ldots, \xi_n, \ldots, u_j, \ldots, \partial^k u_j / \partial \xi_n^{k_0} \ldots \partial \xi_n^{k_n},$$
$$k = \Sigma k_s \leqslant n_j,\ k_0 < n_j \quad (j = 1, 2, \ldots, N).$$

Let us assume that in the neighborhood of the hypersurface $\xi_0 = 0$ on which are prescribed the values of the u_j and of their derivatives we have chosen for $\partial^{n_i}u_i/\partial\xi_0^{n_i}$ $(i=1,2,\dots,N)$ a system of analytic functions in the variables

$$\xi_0,\dots,\xi_n,\dots,u_j,\dots,\frac{\partial^k u_j}{\partial\xi_0^{k_0}\dots\partial\xi_n^{k_n}},\quad k=\Sigma k_s\leqslant n_j, k_0<n_j\quad (j=1,2,\dots,N),$$

satisfying the equations Σ. In this manner we shall have determined the values of $\partial^{n_i}u_i/\partial\xi_0^{n_i}$ on S in accordance with the initial conditions of the generalized Cauchy problem as prescribed on S. If we now revert to the coordinates $x_0,\dots,x_n$, we obtain on S the values of the functions u_i and of all their derivatives with respect to $x_0,\dots,x_n$ of order up to n_i. If these functions are put in place of the u_i and of their derivatives in the equation (12, 3), we obtain a fully determined equation for the $\alpha_0,\alpha_1,\dots,\alpha_n$. This will enable us to determine the characteristic planes at each point $(x_0,x_1,\dots,x_n)$ of the surface S. Let us now assume that S is nowhere characteristic. It can then be shown that in our formulation of the generalized Cauchy problem for the system (11, 3) the prescription of the values of the $\partial^{n_i}u_i/\partial\xi_0^{n_i}$ on S guarantees the existence of a unique analytic solution of the problem.

§ 4. Uniqueness of the solution of the Cauchy problem in the class of non-analytic functions.

1. We know from the Cauchy-Kowalewski theorem that the Cauchy problem has a unique analytic solution provided the analytic surface S on which the analytic initial data are prescribed is nowhere characteristic. It follows from §§ 2 and 3 that, if all functions appearing in the given equations as well as in the initial conditions assume real values for real values of their arguments, then the solutions of the Cauchy problem are also real. We might ask whether in this case the Cauchy problem admits of solutions other than the analytic solution whose existence follows from the Cauchy-Kowalewski theorem. (As a matter of fact, a system $(u_1,\dots,u_N)$ of functions can be a solution of the Cauchy problem without the functions of the system being analytic. For that it suffices that the u_i have derivatives of those orders which appear in the considered equations.) In spite of the efforts of many outstanding mathematicians this question has not as yet been fully answered.

Already in 1901 Holmgren proved the uniqueness of the solution of the Cauchy problem for linear systems of the form (8, 3) with analytic coefficients in the class of functions with continuous derivatives of those orders which appear in the system under consideration. It is also assumed that the analytic surface S (the carrier of the initial data) is nowhere characteristic.

We now give a proof of this theorem.

For simplicity in presentation we assume that the number of independent variables equals two, although no essential modifications of our proof are required if the number of independent variables is greater than two. We also assume that we are dealing with a first-order system. According to § 2, the general case can be reduced to the case of a first-order system. We denote the independent variables by $\tilde{x}$ and y, and at first we assume that the Cauchy problem has been formulated for a segment of the straight line $\tilde{x}=0$ containing the origin.

Thus, let there be given a system of equations

$$\frac{\partial z_i}{\partial \tilde{x}} = \sum_{j=1}^{n} A_{ij}(\tilde{x}, y)\frac{\partial z_j}{\partial y} + \sum_{j=1}^{n} B_{ij}(\tilde{x}, y)\, z_j + C_i(\tilde{x}, y), \tag{1,4}$$

and initial conditions $\qquad z_i(0, y) = \phi_i(y). \qquad (2,4)$

Here A_{ij}, B_{ij} and C_i are analytic functions of their arguments in some neighborhood of the origin. Now let us assume that we are given in some neighborhood of the origin two solutions $z_1, \ldots, z_n$ and $\tilde{z}_1, \ldots, \tilde{z}_n$ of the system (1, 4) satisfying the same initial conditions (2, 4). We assume that the solution functions have continuous derivatives of the first order. We are to show that the two solutions coincide in some neighborhood of the origin.

Let us put $\qquad z_i - \tilde{z}_i = \tilde{u}_i \quad (i = 1, 2, \ldots, n).$

Then all $\tilde{u}_i$ are continuously differentiable in a neighborhood of the origin and satisfy the equations

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}} = \sum_{j=1}^{n} A_{ij}(\tilde{x}, y)\frac{\partial \tilde{u}_j}{\partial y} + \sum_{j=1}^{n} B_{ij}(\tilde{x}, y)\, \tilde{u}_j \quad (i = 1, \ldots, n)$$

and the initial conditions

$$\tilde{u}_i(0, y) = 0 \quad (i = 1, 2, \ldots, n).$$

We shall show that in some neighborhood of the origin $\tilde{u}_i \equiv 0$ for all $\tilde{u}_i$. In place of $\tilde{x}$ we introduce a new independent variable x defined by means of the equation

$$x = \tilde{x} + y^2,$$

and we put

$$u_i(x, y) = \tilde{u}_i(\tilde{x}, y) = \tilde{u}_i(x - y^2, y) \quad (i = 1, \ldots, n).$$

Then the functions u_i satisfy the system of equations

$$\frac{\partial u_i}{\partial x} = 2y \sum_{j=1}^{n} A_{ij}(x-y^2, y)\frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} A_{ij}(x-y^2, y)\frac{\partial u_j}{\partial y} + \sum_{j=1}^{n} B_{ij}(x-y^2, y)\, u_j, \qquad (3,4)$$

or, after solving for the derivatives with respect to x and after introducing the new notations, the system

$$\frac{\partial u_i}{\partial x} = \sum_{j=1}^{n} a_{ij}(x, y)\frac{\partial u_j}{\partial y} + \sum_{j=1}^{n} b_{ij}(x, y)\, u_j. \qquad (4,4)$$

(That it is possible to solve the system (3, 4) for the indicated derivatives follows from the fact that the determinant of the system does not vanish in the neighborhood of the origin of the (x,y) plane.) The coefficients a_{ij} and b_{ij} are analytic in the neighborhood of the origin. The functions u_i are continuously differentiable in the neighborhood of the origin and vanish on the parabola $y^2 = x$. We shall show that in the neighborhood of the origin $u_i \equiv 0$ for all u_i provided that $x > y^2$. It will then follow that in the neighborhood of the origin $\tilde{u}_i \equiv 0$ for all $\tilde{u}_i$ provided that $\tilde{x} > 0$. The case $\tilde{x} < 0$ is reduced to the case $\tilde{x} > 0$ by changing $\tilde{x}$ to $-\tilde{x}$.

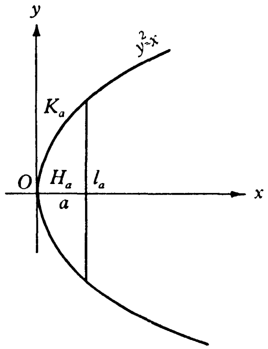

Figure 3

We draw the straight line $x = a$ ($a > 0$; cf. fig. 3) and use the symbol H_a to denote the region bounded by the segment l_a of that straight line and by the portion K_a of the parabola $y^2 = x$. If a is sufficiently small, then all the functions u_i, together with their derivatives of the first order, are continuous up to the boundary of H_a. (We say that a function f defined in a region H is continuous up to the boundary H^* of H if there exists a function g defined and continuous on $H + H^*$ and equal to f on H.)

For the sake of brevity we put

$$F_i(u) \equiv \frac{\partial u_i}{\partial x} - \sum_{j=1}^{n} a_{ij}\frac{\partial u_j}{\partial y} - \sum_{j=1}^{n} b_{ij} u_j \qquad (i = 1, \ldots, n),$$

$$G_i(v) \equiv \frac{\partial v_i}{\partial x} - \sum_{j=1}^{n} \frac{\partial}{\partial y}(a_{ji} v_j) + \sum_{j=1}^{n} b_{ji} v_j \quad (i = 1, \ldots, n).$$

Let there be given on H_a two systems of functions u_i and v_i, continuous together with their first partial derivatives up to the boundary of H_a. Integrating by parts we get

$$\iint_{H_a}\left\{\sum_{i=1}^{n}[v_iF_i(u)+u_iG_i(v)]\right\}dx\,dy$$

$$=\iint_{H_a}\left\{\sum_{i=1}^{n}\left[\frac{\partial(u_iv_i)}{\partial x}-\sum_{j=1}^{n}\frac{\partial(a_{ij}u_jv_i)}{\partial y}\right]\right\}dx\,dy$$

$$=\int_{l_a}\sum_{i=1}^{n}u_iv_i\,dy+\int_{K_a}\sum_{i=1}^{n}u_iv_i\,dy+\int_{K_a}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}u_jv_i\,dx. \qquad (5,4)$$

Here the contour of H_a (i.e. K_a+l_a) is traversed in the positive direction. If, in particular, u_i is the above-defined system of solutions of equation (4,4) and the system of functions $v_1,\dots,v_n$ satisfies the equations

$$G_i(v)=0 \quad (i=1,\dots,n), \qquad (6,4)$$

then, by (5,4)

$$\int_{l_a}\sum_{i=1}^{n}u_iv_i\,dy=0. \qquad (7,4)$$

We now employ the proof of the Cauchy-Kowalewski theorem for a system of linear equations. We use the remark at the conclusion of § 2 and we solve the system (6,4) in the neighborhood of the point $(a,0)$ prescribing as initial conditions on l_a all possible systems of polynomials. If a is small, this remark permits us to conclude that with initial conditions of type mentioned all solutions of the system (6,4) will certainly be analytic in H_a and, consequently, continuous, together with their partial derivatives, up to the boundary of H_a.

Hence, the equality (7,4) holds for $0<a<a_0$ (a_0 is some fixed positive number), and for arbitrary polynomials v_i. We take such arbitrary a and we denote the length of l_a by s_a. By the well-known theorem of Weierstrass, given any $\epsilon>0$ it is possible to find a system of polynomials v_i $(i=1,\dots,n)$ such that at all points of l_a

$$|u_i-v_i|<\epsilon \quad (i=1,\dots,n). \qquad (8,4)$$

In view of (7,4) and (8,4)

$$\int_{l_a}\sum_{i=1}^{n}u_i^2\,dy=\int_{l_a}\sum_{i=1}^{n}u_iv_i\,dy+\int_{l_a}\sum_{i=1}^{n}u_i(u_i-v_i)\,dy\leqslant\epsilon s_a\sum_{i=1}^{n}\max_{l_a}|u_i|,$$

whence, in view of the arbitrariness of ϵ, we have

$$\int_{l_a}\sum_{i=1}^{n}u_i^2\,dy=0,$$

i.e. $u_i\equiv 0$ on l_a if only $0<a<a_0$. This concludes the proof of Holmgren's theorem.

Using this theorem and making a change of the independent variables one can prove easily the uniqueness of the solution of the generalized Cauchy problem for the system (1, 4) provided our previous assumptions about (1, 4) hold and the initial data are prescribed on an analytic curve (or, if $n > 1$, on an analytic surface S). The only additional requirement is that this curve or the surface S be nowhere characteristic for the system (1, 4). The solution functions may be required to exist on one side of the surface S only and to be continuous up to S together with their partial derivatives of the first order. The uniqueness theorem remains true in the following formulation: If two solutions with continuous first derivatives satisfy all the conditions prescribed for them and coincide on S, then they must necessarily coincide in some neighborhood of S.

Remark. It is possible to give a more precise description of the region in the $(\tilde{x}, y)$ plane in which the solution of the Cauchy problem for the system (1, 4) is uniquely determined by the initial conditions (the domain of determinacy). Thus, let the initial data be prescribed on a segment AB of the y-axis. Through A and B we draw the characteristics which are closest to AB. It is then possible to show that the domain of determinacy of the solution of the Cauchy problem is the region bounded by AB and by these two characteristics. An analogous determination of the domain of determinacy holds for a greater number of independent variables (cf. §§ 10 and 12 where the Cauchy problem is solved for hyperbolic equations).

2. Soon after the proof of Holmgren's theorem had been given, Hadamard showed that the problem of uniqueness in the neighborhood of S of the solution of the Cauchy problem for non-linear equations could be easily reduced to the problem of uniqueness of the solution of the Cauchy problem for linear equations with sufficiently smooth but not necessarily analytic coefficients. All further efforts centered around the solution of the latter problem. In 1938 Carleman solved this problem for a system of partial differential equations in two independent variables. Carleman's theorem can be stated as follows:

Let there be given a system of equations

$$\frac{\partial z_i}{\partial x} + \sum_{j=1}^{n} A_{ij}(x, y) \frac{\partial z_j}{\partial y} + B_{ij}(x, y) z_j = 0 \quad (i = 1, 2, \ldots, n). \qquad (9, 4)$$

The functions A_{ij} and B_{ij} are prescribed in some closed region $\bar{G}$ of the half-plane $x \geqslant 0$ abutting on the segment $|y| \leqslant a$ of the axis of ordinates. The A_{ij} have in $\bar{G}$ bounded derivatives of order up to and including the second. The B_{ij} are bounded and continuous in $\bar{G}$. Then the solution in $\bar{G}$ of the system (9, 4) satisfying the conditions

$$z_i(0, y) = 0 \text{ with } |y| \leqslant a \quad (i = 1, \dots, n)$$

and having continuous first derivatives with respect to x and y vanishes identically in some portion $\bar{G}'$ of $\bar{G}$ abutting on the segment $|y| \leqslant a$. It is assumed that at each point of $\bar{G}$ all roots of the determinant

$$|A_{ij} - \lambda\delta_{ij}| \quad (\delta_{ij} = 1 \text{ if } i = j;\ \delta_{ij} = 0 \text{ if } i \neq j),$$

are distinct, i.e. that at no point of $\bar{G}$ do two characteristic directions coincide. As was shown in 1947 by A. Myshkis, if these conditions are violated the uniqueness assertion may not hold. As an example Myshkis considered the system

$$\frac{\partial u}{\partial x} = a_1(x, y)\frac{\partial u}{\partial y} + b_1(x, y)\frac{\partial v}{\partial y},$$

$$\frac{\partial v}{\partial x} = a_2(x, y)\frac{\partial u}{\partial y} + b_2(x, y)\frac{\partial v}{\partial y},$$

with coefficients prescribed and differentiable in the whole plane and with some solution u_0, v_0. The functions u_0 and v_0 have continuous partial derivatives of all orders; they vanish on the straight line $x = 0$ but differ from zero at points arbitrarily close to the origin. The derivatives of the coefficients of the system are discontinuous on the y-axis, and the roots of the characteristic equation coincide there. It is not known whether the assertion concerning the uniqueness of the solution of the Cauchy problem remains valid if the derivatives of the coefficients are continuous.†

As already mentioned the problem of uniqueness for equations and systems with non-analytic coefficients and an arbitrary number of independent variables has not as yet been solved. The uniqueness of the solution of the Cauchy problem in the class of sufficiently smooth functions has been proved for so-called hyperbolic systems which will be discussed later.

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† See the article by A. Myshkis in *Uspekhi matematicheskikh nauk*, vol. III (2nd ed., 1948), pp. 3–46. In it are formulated all essential facts connected with the question of uniqueness of the solution of the Cauchy problem. The proofs of the more important of these facts are given. The article also contains a detailed bibliography on this question.

The problem of uniqueness of the solution of the Cauchy problem in the domain of non-analytic but sufficiently smooth functions is connected with the problem of the possibility of continuing in a unique way a sufficiently smooth real solution $(u_1, \ldots, u_N)$ of the system (11, 3) of the preceding section, given in some real region in the space $(x_0, \ldots, x_n)$ on one side of a sufficiently smooth and nowhere characteristic surface S and on S itself. In fact, prescribing the functions u_i on one side of the surface S and on the surface S itself determines the values on S of the u_i and their derivatives which appear in the Cauchy conditions. Thus, the problem of continuing the functions u_i to the other side of the surface S is reduced to that of finding a solution of the generalized Cauchy problem in the region lying on the other side of the surface S. As mentioned above, the problem of uniqueness of this solution has not as yet been completely solved.

The following is another question which has not yet been completely answered. Let $(u_1, \ldots, u_N)$ be a sufficiently smooth solution of the system (11, 3) given on a sufficiently smooth surface S and on one side of S. Can this solution be continued in different ways if the surface S is characteristic for the given system and the given solution? As for the equations with which we shall be concerned, it will always be possible to continue a solution in a great many ways.

The question of non-unique continuation of a solution of the system (11, 3) beyond a real characteristic is equivalent to the question of existence of many solutions of the generalized Cauchy problem provided the Cauchy conditions prescribed on the characteristic admit at least one such solution. We have seen that for this the functions u_i and their derivatives, prescribed on the characteristic, must, in general, satisfy certain relations. These relations are certainly satisfied if there exist functions $u_1, \ldots, u_N$ satisfying the given equations on one or the other side of the real characteristic.

If we restrict our attention to analytic solutions only, then the continuation of a solution given on an $n+1$-dimensional region beyond a characteristic, or, more generally, beyond any surface, is always unique, for an analytic function of $n+1$ independent variables is fully determined by its values in an arbitrarily small region of dimension $n+1$.

3. We have seen in para. 3, § 3, that if the surface S on which are prescribed Cauchy conditions is nowhere characteristic, then these

conditions, together with the equations of the system (6, 3), determine uniquely on S the values of the u_i and their derivatives of order up to n_i. On the other hand, if the surface S is characteristic in the neighborhood of a point A, then the Cauchy conditions prescribed on S admit various systems of values $\partial^{n_i} u_i/\partial\xi_0^{n_i}$ which satisfy the system (6, 3) (it is assumed that $\xi_0 = 0$ is the equation of the surface S). Therefore, there may exist functions satisfying (6, 3) at all points of a region containing a piece of the characteristic surface whose derivatives $\partial^{n_i} u_i/\partial\xi_0^{n_i}$ have discontinuities of the first kind on this surface. Depending on the direction of approach to S, these derivatives approach different values satisfying simultaneously equations (6, 3) on S. Suppose S were not characteristic. Then continuity of the coefficients of equations (6, 3) and of all other derivatives of order n_i of the functions u_i would preclude the discontinuity of the derivatives $\partial^{n_i} u_i/\partial\xi_0^{n_i}$ on S. An analogous statement is valid for non-linear systems.

Example. Consider the equation

$$\partial^2 u/\partial x \partial y = 0 \tag{10,4}$$

for which the lines $x = \text{const.}$, $y = \text{const.}$ are characteristics.

Clearly, (10, 4) is satisfied by any everywhere differentiable function

$$u = f(y).$$

In particular we may assume that $u = f(y)$ is such that its second derivative is continuous everywhere with the exception of one point at which it has a discontinuity of the first kind. We then obtain a solution of the equation (10, 4) whose second partial derivatives have a discontinuity of the first kind on the characteristic.

In what follows we shall concentrate on equations of two types. To the first type belong single equations of the *second* order in one unknown function. To the second type belong systems of any order in any number of unknown functions but with derivatives with respect to *two* independent variables only. Equations of these two types can be reduced to certain simple 'canonical' forms. This reduction process is described in the following three sections.

§ 5. Reduction to canonical form at a point and classification of equations of the second order in one unknown function

1. We consider the linear, second-order equation

$$\sum_{i,j} A_{ij}(x_1, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i B_i(x_1, \ldots, x_n) \frac{\partial u}{\partial x_i} + C(x_1, \ldots, x_n) u + F(x_1, \ldots, x_n) = 0 \quad (1, 5)$$

in one unknown function u. We assume that $A_{ij} = A_{ji}$. A_{ij}, B_{ij}, C and F are all real-valued and defined in some region of the space $(x_1, \ldots, x_n)$. We now define a coordinate transformation by putting

$$\xi_k = \sum_{i=1}^{n} a_{ki} x_i \quad (k = 1, \ldots, n), \qquad (2, 5)$$

where a_{ki} are certain constants. We assume that the transformation (2, 5) is non-singular, i.e. that the determinant $|a_{ki}|$ does not vanish. Then the transformation (2, 5) is one-to-one. In the new coordinates $\xi_1, \xi_2, \ldots, \xi_n$, equation (1, 5) will take the form

$$\sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} A_{ij} a_{ki} a_{lj} \right) \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} + \ldots = 0.\dagger \qquad (3, 5)$$

Here we have written out only terms with second-order derivatives of the unknown function u. It is clear from the equality (3, 5) that under the transformation (2, 5) of the independent variables the coefficients of the second-order derivatives of u change in exactly the same manner in which the coefficients of the quadratic form

$$\sum_{i,j=1}^{n} A_{ij} x_i x_j \qquad (4, 5)$$

change when the x_k are replaced by the ξ_k, according to the formulas

$$x_k = \sum_{i=1}^{n} a_{ik} \xi_i \quad (k = 1, \ldots, n). \qquad (5, 5)$$

We regard the coefficients A_{ij} of the form (4, 5) as constants whose values are the same as those of the values of the coefficients $A_{ij}(x_1, \ldots, x_n)$ of equation (1, 5) at some point $(x_1^0, \ldots, x_n^0)$ in G.

It is proved in algebra that there exists a real non-singular transformation (5, 5) which reduces any form (5, 4) with real coeffi-

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† For the formal transition from the derivatives with respect to the independent variables x_i $(i = 1, \ldots, n)$ to the derivatives with respect to the independent variables ξ_i $(i = 1, \ldots, n)$ to be correct, it suffices to assume that the function u is twice continuously differentiable.

cients A_{ij} to the form

$$\sum_{i=1}^{m} \pm \xi_i^2, \quad \text{where} \quad m \leqslant n. \tag{6,5}$$

There are many non-singular real transformations (5, 5) which reduce the form (4, 5) to the form (6, 5), but the number of terms with positive signs and the number of terms with negative signs in the form (6, 5) is determined entirely by the form (4, 5) and is independent of the choice of the non-singular transformation (5, 5). (Law of inertia of quadratic forms.†)

If a certain transformation (5, 5) reduces the form (4, 5) to the form (6, 5), then the transformation (2, 5) with matrix equal to the transposed inverse of (a_{ik}) reduces (1, 5) to the form

$$\sum_{i,j=1}^{n} A_{ij}^*(x_1, \ldots, x_n) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \ldots = 0, \tag{7,5}$$

where
$$A_{ij}^*(x_1^0, \ldots, x_n^0) = \pm 1 \quad \text{if } i = j \leqslant m,$$
$$A_{ij}^*(x_1^0, \ldots, x_n^0) = 0 \quad \text{if } i \neq j \text{ or if } i = j > m.$$

Here we have written out only terms containing derivatives of the highest order of the function u. The form (7, 5) of equation (1, 5) is called its *canonical form at the point* $(x_1^0, \ldots, x_n^0)$.

It is thus possible to exhibit for each point $(x_1^0, \ldots, x_n^0)$ of the region G a non-singular transformation of the independent variables which reduces equation (1, 5) at that point to canonical form.

In general, the transformation (2, 5) which reduces equation (1, 5) to canonical form at a given point $(x_1^0, \ldots, x_n^0)$ varies with this point, i.e. it may not reduce equation (1, 5) to canonical form at a point different from $(x_1^0, \ldots, x_n^0)$. It can be shown by means of examples that as soon as the number of independent variables exceeds two, it is, in general, impossible to exhibit a linear transformation with constant coefficients, or, for that matter, any other transformation which would reduce a given linear equation of the second order to canonical form in an *arbitrarily small region*. In the case of two variables such a transformation exists under very general assumptions on the coefficients of equation (1, 5). This fact will be demonstrated in the following section.

The classification of equations of the second order is based on the possibility of reducing equation (1, 5) to canonical form at a point.

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† See G. Birkhoff and S. MacLane, *A Survey of Modern Algebra* (1951), Macmillan New York.

2. Equation (1, 5) is said to be *elliptic* at a point $(x_1^0, \ldots, x_n^0)$ if all $A_{ii}^*(x_1^0, \ldots, x_n^0)$ in (7, 5) are different from zero and have the same sign.

Equation (1, 5) is said to be *hyperbolic* at a point $(x_1^0, \ldots, x_n^0)$ if all but one $A_{ii}^*(x_1^0, \ldots, x_n^0)$ in (7, 5) have the same sign, the exceptional A_{ii}^* is of opposite sign and $m = n$.

Equation (1, 5) is said to be *ultrahyperbolic* at a point $(x_1^0, \ldots, x_n^0)$ if in (7, 5) the number of positive $A_{ii}^*(x_1^0, \ldots, x_n^0)$ exceeds one, the number of negative $A_{ii}^*(x_1^0, \ldots, x_n^0)$ exceeds one, and $m = n$.

Equation (1, 5) is said to be *parabolic in the broad sense* at a point $(x_1^0, \ldots, x_n^0)$ if some $A_{ii}^*(x_1^0, \ldots, x_n^0)$ are zero, i.e. if $m < n$.

Equation (1, 5) is said to be *parabolic in the restricted sense* or, simply, *parabolic* if only one of the coefficients $A_{ii}^*(x_1^0, \ldots, x_n^0)$ (A_{11}^*, say) is zero and all other A_{ii}^* have the same sign, and the coefficient of $\partial u/\partial \xi_1$ is different from zero.

Equation (1, 5) is said to be *elliptic, hyperbolic, ultrahyperbolic*, etc., in the whole region G if it is elliptic, hyperbolic, ultrahyperbolic, etc., at each point of the region G.

3. The non-linear equation of the second order

$$\Phi\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \ldots, \frac{\partial^2 u}{\partial x_i \partial x_j}, \ldots\right) = 0$$

in one unknown function u is said to be elliptic, hyperbolic or parabolic in the broad sense *for a given solution* $u^*(x_1, \ldots, x_n)$ at a point $(x_1^0, \ldots, x_n^0)$ (or in the region G) if the equation

$$\sum_{i,j=1}^{N} A_{ij}(x_1, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

where

$$A_{ij}(x_1, \ldots, x_n) = \frac{\partial \Phi}{\partial \left\{\dfrac{\partial^2 u}{\partial x_i \partial x_j}\right\}} \tag{8, 5}$$

is elliptic, hyperbolic or parabolic in the broad sense at the point $(x_1^0, \ldots, x_n^0)$ (or in G). On the right side of (8, 5) the function u and its derivatives are replaced by the function $u^*(x_1, \ldots, x_n)$ and its appropriate derivatives.

Later we shall study only linear equations of the second order in one unknown function which are either elliptic, or hyperbolic or parabolic in the whole region under consideration. We shall not be interested in ultrahyperbolic equations, for they do not appear in physics or in engineering. We shall, likewise, take no interest in equations which are parabolic in the broad sense, for they turn up seldom in applied work. When speaking of parabolic equations in Chapter IV we shall, accordingly, have in mind equations parabolic in the narrow sense.

§ 6. Reduction to canonical form in a region of a partial differential equation of the second order in two independent variables

1. Consider the equation

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0. \qquad (1,6)\dagger$$

Here the coefficients A, B and C are functions of x and y which are twice continuously differentiable. We shall assume that the function $u(x,y)$ is also twice continuously differentiable. We now go from the independent variables x, y to the independent variables ξ, η. Let the functions

$$\xi=\xi(x,y),\quad \eta=\eta(x,y) \qquad (2,6)$$

be twice continuously differentiable, and let the Jacobian

$$\begin{vmatrix}\dfrac{\partial\xi}{\partial x} & \dfrac{\partial\xi}{\partial y}\\[2ex] \dfrac{\partial\eta}{\partial x} & \dfrac{\partial\eta}{\partial y}\end{vmatrix}$$

be different from zero in the region under consideration. It is then possible to solve the system (2, 6) uniquely for x and y in some region of the (ξ, η) plane. The resulting functions $x(\xi, \eta)$ and $y(\xi, \eta)$ will also be twice continuously differentiable functions of ξ and η. In the new independent variables, equation (1, 6) takes the form

$$\begin{aligned}&\frac{\partial^2 u}{\partial\xi^2}\left[A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2\right]\\ &\quad+2\frac{\partial^2 u}{\partial\xi\,\partial\eta}\left[A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+B\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right]\\ &\quad+\frac{\partial^2 u}{\partial\eta^2}\left[A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2\right]+F_1\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0.\end{aligned} \qquad (3,6)$$

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† In this section we consider equations which are somewhat more general than linear equations, thus utilizing the fact that the procedure employed for the purpose of reducing a linear equation to canonical form is also applicable to such equations.

We now consider the differential equation

$$A\left(\frac{\partial\phi}{\partial x}\right)^2+2B\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}+C\left(\frac{\partial\phi}{\partial y}\right)^2=0 \tag{4,6}$$

in the unknown function $\phi(x,y)$. We shall have to deal separately with the cases $B^2>AC$, $B^2<AC$ and $B^2=AC$ in all of the region under consideration. We shall not consider the case when the expression B^2-AC changes sign in the region considered or when it vanishes at some point of the region without vanishing identically in all of it.

2. We first investigate the case $B^2>AC$ (in the region under consideration), i.e. the case when equation (1, 6) is hyperbolic (cf. the definition of hyperbolicity given in the preceding paragraph).

In this case equation (4, 6) is equivalent to the two equations

$$\left.\begin{aligned}A\frac{\partial\phi_1}{\partial x}&=[-B-(B^2-AC)^{\frac{1}{2}}]\frac{\partial\phi_1}{\partial y},\\ A\frac{\partial\phi_2}{\partial x}&=[-B+(B^2-AC)^{\frac{1}{2}}]\frac{\partial\phi_2}{\partial y}.\end{aligned}\right\} \tag{5,6}$$

If the coefficient A vanishes, then one of the equations (5, 6) becomes indeterminate (if, for instance, $B>0$, then the second equation in (5, 6) becomes indeterminate). Its indeterminacy, however, is easily removed by multiplying both sides by the identically equal expressions

$$[-B-(B^2-AC)^{\frac{1}{2}}]/A\equiv C/[-B+(B^2-AC)^{\frac{1}{2}}]. \tag{6,6}$$

We now determine the functions $\phi_i(x,y)$ $(i=1,2)$ as solutions of the equations (5, 6) by prescribing their values on certain curves l_i which are nowhere tangent to the characteristics of the corresponding equation.† If the curves l_i and the values of the functions ϕ_i prescribed on them are sufficiently smooth, we obtain solution functions $\phi_i(x,y)$ $(i=1,2)$ having continuous derivatives with respect to x and y of order up to and including the second. If, in addition, we assume that the initial values $\phi_i(x,y)$ on l_i have been chosen so that the derivative of ϕ_i in the direction of l_i does not vanish anywhere, then at no point will the derivatives of the functions $\phi_i(x,y)$ with respect to x and y vanish simultaneously. Since the vanishing at some point of these two derivatives would imply the vanishing at that point of the derivative of ϕ_i in an arbitrary direction, our statement will have been proved if we can show that at no point A does the derivative of $\phi_i(x,y)$ vanish in all directions. Let us move in a non-characteristic direction from the point A to some point A'. Let us denote by B the point of intersection of l_i and the characteristic through A, and by B' the point of intersection of l_i and the characteristic through A' (fig. 4). Then, in view of the fact that $\phi_i(x, y)$ is constant along a characteristic, we have

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† Cf. § 52 of my *Lectures on the Theory of Ordinary Differential Equations.* I call the reader's attention to the fact that in the case of two independent variables the definition of characteristics as given in the above work coincides with the definition of characteristics introduced in §3. For a larger number of independent variables, however, the two definitions are dissimilar.

Cf. also Courant-Hilbert, *Methods of Mathematical Physics*, vol. II, Interscience, New York–London.

$$\phi_i(A)=\phi_i(B),$$

$$\phi_i(A')=\phi_i(B'),$$

$$\phi_i(A')-\phi_i(A)=\phi_i(B')-\phi_i(B).$$

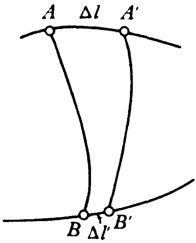

Figure 4

The two distances, Δl between the point A and A' and $\Delta l'$ between the points B and B', approach zero simultaneously. Also, in view of the assumed smoothness of the coefficients of equation (1, 6), these two distances are of the same order of smallness. Since, by assumption,

$$\lim_{\Delta l'\to 0} \frac{\phi_i(B')-\phi_i(B)}{\Delta l'} \neq 0,$$

we must also have $$\lim_{\Delta l\to 0} \frac{\phi_i(A')-\phi_i(A)}{\Delta l} \neq 0.$$

It follows that at no point do the derivatives with respect to x and y of either one of the functions $\phi_1(x,y)$ and $\phi_2(x,y)$ vanish simultaneously. This means that the Jacobian

$$\begin{vmatrix} \dfrac{\partial\phi_1}{\partial x} & \dfrac{\partial\phi_1}{\partial y} \\ \dfrac{\partial\phi_2}{\partial x} & \dfrac{\partial\phi_2}{\partial y} \end{vmatrix}$$

of these functions cannot vanish at any point of the region under consideration; for, since neither row of the Jacobian vanishes, a necessary condition for its vanishing is the proportionality of its columns, but

$$\frac{\partial\phi_1}{\partial x}:\frac{\partial\phi_1}{\partial y}=\frac{-B-(B^2-AC)^{\frac{1}{2}}}{A}\neq\frac{-B+(B^2-AC)^{\frac{1}{2}}}{A}=\frac{\partial\phi_2}{\partial x}:\frac{\partial\phi_2}{\partial y},$$

because $B^2-AC\neq 0$. An indeterminacy in any one of our relations can be removed by using the identity (6, 6).

Since the Jacobian of ϕ_1 and ϕ_2 with respect to x and y does not vanish, we may put in (2, 6)

$$\left.\begin{aligned}\xi=\xi(x,y)=\phi_1(x,y),\\ \eta=\eta(x,y)=\phi_2(x,y).\end{aligned}\right\} \tag{7, 6}$$

If we do this, the terms in (3, 6) which contain $\partial^2u/\partial\xi^2$ and $\partial^2u/\partial\eta^2$ must vanish. At the same time, the coefficient of $\partial^2u/\partial\xi\partial\eta$ *will be different from zero* in the whole region considered. Otherwise, a change from the coordinates (x,y) to the coordinates (ξ,η) would decrease the order of the equation, and, consequently, the opposite change, from the coordinates (ξ,η) to the coordinates (x,y) would increase the order of the equation, which is an obvious impossibility.

Dividing equation (3, 6) by the coefficient of $\partial^2u/\partial\xi\partial\eta$ we reduce it to the canonical form

$$\frac{\partial^2u}{\partial\xi\partial\eta}=F_2\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right) \tag{8, 6}$$

in all of the region of definition of the functions $\phi_1(x,y)$ and $\phi_2(x,y)$. If we put $\xi=\alpha+\beta$ and $\eta=\alpha-\beta$, equation (8, 6) becomes

$$\frac{\partial^2u}{\partial\alpha^2}-\frac{\partial^2u}{\partial\beta^2}=\Phi\left(\alpha,\beta,u,\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial\beta}\right). \tag{9, 6}$$

The latter form of the equation is also called canonical.

If a hyperbolic equation has been reduced to the canonical form (8, 6), it is sometimes possible to integrate it in closed form, i.e. to find a formula giving all solutions of this equation.

Example 1. By means of the substitution

$$\partial u/\partial\eta=v,$$

the equation

$$\frac{\partial^2u}{\partial\xi\partial\eta}=\frac{1}{2\xi}\frac{\partial u}{\partial\eta} \tag{10, 6}$$

becomes $$\partial v/\partial \xi = v/(2\xi).$$

The latter equation can be easily integrated by the method of separation of variables, for η enters in v only as a parameter. The constant of integration will be a function of this parameter. We have

$$\ln v = \tfrac{1}{2}\ln \xi + \ln C(\eta)$$

or $$v = \partial u/\partial \eta = C(\eta)\,(\xi)^{\frac{1}{2}}.$$

Whence $$u = C_1(\eta)\,(\xi)^{\frac{1}{2}} + C_2(\xi).$$

Here $$C_1(\eta) = \int C(\eta)\,d\eta$$

is an arbitrary (in view of the arbitrariness of $C(\eta)$) differentiable function of η and $C_2(\xi)$ is an arbitrary function of ξ.

Example 2. The transformation

$$x = \xi + \eta, \quad y = \xi - \eta$$

reduces the equation $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{11, 6}$$

to the form $$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0. \tag{12, 6}$$

Hence $$u = \phi(\xi) + \psi(\eta) = \phi_1(x+y) + \psi_1(x-y), \tag{13, 6}$$

where ϕ and ψ are arbitrary twice continuously differentiable functions.

3. If $$B^2 = AC$$

in all of the region G, then equation (1, 6) is parabolic in G (cf. the definition of parabolicity in the preceding section). In this case both equations (5, 6) are the same and can be replaced by the single equation

$$A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y} = 0, \tag{14, 6}$$

or, equivalently, by the equation

$$B\frac{\partial \phi}{\partial x} + C\frac{\partial \phi}{\partial y} = 0. \tag{15, 6}$$

Let $$\phi(x, y) = C$$

be the general integral of this equation in some region G of the (x,y) plane. We assume that the function $\phi(x,y)$ is twice continuously differentiable and that its first partial derivatives do not vanish simultaneously (cf. para. 2). Let

$$\psi(x,y)=C$$

be a family of curves in the region G such that the function $\psi(x,y)$ is sufficiently smooth and the Jacobian

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial\psi}{\partial x} & \dfrac{\partial\psi}{\partial y} \end{vmatrix} \tag{16, 6}$$

does not vanish anywhere in the region G. If, for example,

$$\partial\phi/\partial x>0$$

at all points of the region G, we may assume that

$$\psi(x,y)=y.$$

Let us put $\quad \xi=\phi(x,y) \quad$ and $\quad \eta=\psi(x,y)$

in (2, 6). Then the coefficient of $\partial^2u/\partial\xi^2$ in (3, 6) vanishes. The coefficient of $\partial^2u/\partial\xi\,\partial\eta$ becomes

$$\left(A\frac{\partial\phi}{\partial x}+B\frac{\partial\phi}{\partial y}\right)\frac{\partial\phi}{\partial x}+\left(B\frac{\partial\phi}{\partial x}+C\frac{\partial\phi}{\partial y}\right)\frac{\partial\psi}{\partial y}.$$

According to (14, 6) and (15, 6), this coefficient will also vanish. We assume that at no point of the region G do all three coefficients A, B and C of equation (1, 6) vanish simultaneously. Since $B^2=AC$, this assumption implies that at each point of G one of the coefficients A and C is different from zero. Let $A\neq 0$ at some definite point. Then the coefficient of $\partial^2u/\partial\eta^2$ in equation (3, 6) takes the form

$$A\left(\frac{\partial\psi}{\partial x}\right)^2+2B\frac{\partial\psi}{\partial x}\cdot\frac{\partial\psi}{\partial y}+C\left(\frac{\partial\psi}{\partial y}\right)^2=\frac{1}{A}\left(A\frac{\partial\psi}{\partial x}+B\frac{\partial\psi}{\partial y}\right)^2.$$

This expression cannot be zero: for, otherwise, in view of (14, 6), the Jacobian (16, 6) would vanish at the point under consideration. We can show in the same way that the coefficient of $\partial^2u/\partial\eta^2$ does not vanish at points at which $C\neq 0$. This means that this coefficient cannot vanish anywhere in the region G. Consequently, we are allowed to divide equation (3, 6) by this coefficient. We then obtain

$$\frac{\partial^2 u}{\partial \eta^2}+F_2\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right)=0. \tag{17, 6}$$

According to the definition of canonical form given in § 5, equation (17, 6) is in canonical form in the region G.

If equation (1, 6) is linear, then so is equation (17, 6). Suppose that (17, 6) is of the form

$$\frac{\partial^2 u}{\partial \eta^2}=A_1\frac{\partial u}{\partial \xi}+B_1\frac{\partial u}{\partial \eta}+C_1 u+D_1. \tag{18, 6}$$

It is possible to simplify this equation somewhat by introducing a new unknown function z in place of u. We put

$$u=zv,$$

where $v(\xi,\eta)$ is a function of ξ and η still to be defined. Then equation (18, 6) becomes

$$v\frac{\partial^2 z}{\partial \eta^2}+2\frac{\partial v}{\partial \eta}\frac{\partial z}{\partial \eta}=A_1 v\frac{\partial z}{\partial \xi}+B_1 v\frac{\partial z}{\partial \eta}+C_2 z+D_1. \tag{19, 6}$$

We have written out in detail only terms containing derivatives of z. All terms containing the function z itself have been combined in $C_2 z$. We choose the function $v(\xi,\eta)$ so that the coefficient of the derivative $\partial z/\partial\eta$ in equation (19, 6) should vanish. Equating the coefficient of $\partial z/\partial\eta$ to zero, we get

$$\frac{\partial^2 z}{\partial \eta^2}=A_1\frac{\partial z}{\partial \xi}+C_3 z+D_2. \tag{20, 6}$$

where $C_3=C_2/v$, $D_2=D_1/v$, and v is

$$v(\xi,\eta)=\exp\left\{\frac{1}{2}\int B_1(\xi,\eta)\,d\eta\right\}.$$

4. Finally, we investigate the case when in all of the region considered

$$AC>B^2.$$

Then equation (1, 6) will be elliptic in that region (cf. the definition of ellipticity given in § 5). In this case we assume that the coefficients A, B and C are analytic functions of x and y. Then the coefficients of equations (5, 6) are also analytic functions of x and y. Let

$$\phi(x,y)=\phi^*(x,y)+i\phi^{**}(x,y)$$

be an analytic solution of one of the equations in (5, 6) in a neighborhood of a point (x_0, y_0).† We put

$$\xi = \phi^*(x, y) \quad \text{and} \quad \eta = \phi^{**}(x, y) \tag{21, 6}$$

in (2, 6). Since the Jacobian

$$\begin{vmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{vmatrix} \tag{22, 6}$$

does not vanish anywhere, we can solve equations (21, 6) for x and y. In fact, separating real and imaginary parts in the second equation (5, 6), we get

$$\left.\begin{aligned} A\frac{\partial \xi}{\partial x} &= -B\frac{\partial \xi}{\partial y} - (AC - B^2)^{\frac{1}{2}}\frac{\partial \eta}{\partial y}, \\ A\frac{\partial \eta}{\partial x} &= -B\frac{\partial \eta}{\partial y} + (AC - B^2)^{\frac{1}{2}}\frac{\partial \xi}{\partial y}. \end{aligned}\right\} \tag{23, 6}$$

Substituting the expressions for $\partial\xi/\partial x$ and $\partial\eta/\partial x$ obtained from (23, 6) in the Jacobian (22, 6), we obtain

$$-\frac{(AC - B^2)^{\frac{1}{2}}}{A}\left[\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right].$$

It follows that our determinant can be equal to zero only at those points at which

$$\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial y} = 0,$$

or, in view of equations (23, 6), only at those points at which

$$\frac{\partial \xi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 0.$$

But our region contains no such points, for at such points

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0.$$

Separating real and imaginary parts in the identity

$$A\left(\frac{\partial \phi}{\partial x}\right)^2 + 2B\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} + C\left(\frac{\partial \phi}{\partial y}\right)^2 = 0,$$

---

† Let (x_0, y_0) be a point of our region. In some neighborhood of (x_0, y_0) it is possible to find an analytic solution $\phi(x, y)$ of the equation (5, 6) such that $\partial\phi/\partial x$ and $\partial\phi/\partial y$ do not vanish simultaneously at any of the points of that neighborhood. This can be done, for instance, according to the Cauchy-Kowalewski theorem by prescribing the values of $\phi(x, y)$ for $x = x_0$ so that $\phi_y'(x_0, y_0) \neq 0$. We assume that $\phi(x, y)$ is a solution of just this kind.

we get

$$A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2=A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2, \tag{24, 6}$$

$$A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=0. \tag{25, 6}$$

Since the form

$$A\alpha^2+2B\alpha\beta+C\beta^2 \quad (B^2-AC<0)$$

is positive-definite, both sides of (24, 6) can vanish only if

$$\frac{\partial\xi}{\partial x}=\frac{\partial\xi}{\partial y}=\frac{\partial\eta}{\partial x}=\frac{\partial\eta}{\partial y}=0. \tag{26, 6}$$

We chose the function $\phi(x, y)$ in such a manner, however, that the equations (26, 6) are not satisfied at the same time. Hence, we may divide equation (3, 6) by one side of the equation (24, 6). We get

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F_2\left(\xi,\eta,u,\frac{\partial u}{\partial\xi}\frac{\partial u}{\partial\eta}\right). \tag{27, 6}$$

This form of the elliptic equation is called its canonical form.

We have reduced our equation to canonical form in a neighborhood of (x_0, y_0) in which there exists an analytic solution of equation (5, 6) with non-zero derivatives. It can be shown by means of more intricate considerations that such reduction is possible without assuming that $A(x,y)$, $B(x,y)$ and $C(x,y)$ are analytic. It suffices to assume that these coefficients are twice continuously differentiable.

§ 7. Reduction to canonical form of a system of linear partial differential equations of the first order in two independent variables

We consider the system of equations

$$\frac{\partial u_i}{\partial x}=\sum_{j=1}^{n}a_{ij}(x,y)\frac{\partial u_j}{\partial y}+f_i(x,y,u_1,\ldots,u_n) \quad (i=1,2,\ldots,n). \tag{1, 7}$$

Whether or not the f_i are linear with respect to $u_1, u_2, \ldots, u_n$ is of no importance to us. We assume that the coefficients $a_{ij}(x,y)$ have continuous partial derivatives with respect to x and y of all orders up to and including k ($k \geqslant 1$) in some region G in the (x,y) plane. If, in addition to our assumption, certain additional assumptions (appearing below in italics) are made, then it is possible to reduce the system (1, 7) to a *canonical form* in some neighborhood of an arbitrary interior point A of G. The reduction can be effected by means of a linear transformation of the unknown functions $u_1, \ldots, u_N$ with coefficients having the same number of continuous derivatives as the coefficients $a_{ij}(x, y)$. The canonical form to which our system can be reduced is as follows:

$$\left.\begin{aligned}
&\frac{\partial v_1}{\partial x} = \lambda_1(x,y)\frac{\partial v_1}{\partial y} && + f_1^*(x,y,v_1,\ldots,v_n),\\
&\frac{\partial v_2}{\partial x} = \alpha_1(x,y)\frac{\partial v_1}{\partial y} + \lambda_1(x,y)\frac{\partial v_2}{\partial y} && + f_2^*(x,y,v_1,\ldots,v_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial v_{n_1}}{\partial x} = \alpha_{n_1-1}(x,y)\frac{\partial v_{n_1-1}}{\partial y} + \lambda_1\frac{\partial v_{n_1}}{\partial y} && + f_{n_1}^*(x,y,v_1,\ldots,v_n),\\
&\frac{\partial v_{n_1+1}}{\partial x} = \lambda_2(x,y)\frac{\partial v_{n_1+1}}{\partial y} && + f_{n_1+1}^*(x,y,v_1,\ldots,v_n),\\
&\frac{\partial v_{n_1+2}}{\partial x} = \beta_1(x,y)\frac{\partial v_{n_1+1}}{\partial y} + \lambda_2(x,y)\frac{\partial v_{n_1+2}}{\partial y}\\
& && + f_{n_1+2}^*(x,y,v_1,\ldots,v_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial v_{n_1+n_2}}{\partial x} = \beta_{n_2-1}(x,y)\frac{\partial v_{n_1+n_2-1}}{\partial y} + \lambda_2(x,y)\frac{\partial v_{n_1+n_2}}{\partial y}\\
& && + f_{n_1+n_2}^*(x,y,v_1,\ldots,v_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial v_{n-n_k+1}}{\partial x} = \lambda_k(x,y)\frac{\partial v_{n-n_k+1}}{\partial y} && + f_{n-n_k+1}^*(x,y,v_1,\ldots,v_n),\\
&\frac{\partial v_{n-n_k+2}}{\partial x} = \omega_1(x,y)\frac{\partial v_{n-n_k+1}}{\partial y} && + \lambda_k(x,y)\frac{\partial v_{n-n_k+2}}{\partial y}\\
& && + f_{n-n_k+2}^*(x,y,v_1,\ldots,v_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial v_n}{\partial x} = \omega_{n_k-1}(x,y)\frac{\partial v_{n-1}}{\partial y} + \lambda_k(x,y)\frac{\partial v_n}{\partial y}\\
& && + f_n^*(x,y,v_1,\ldots,v_n).
\end{aligned}\right\}\quad (2,7)$$

Here $\lambda_1(x,y), \ldots, \lambda_k(x,y)$ are the roots of the determinant of the matrix

$$\|a_{ij}(x,y)\| - \lambda E, \qquad (3,7)$$

and $\alpha_i(x,y)$, $\beta_i(x,y), \ldots, \omega_i(x,y)$ are essentially arbitrary functions which have continuous derivatives of all orders up to and including k and do not vanish in the considered neighborhood of the point A. In general, the functions $v_i, \lambda_i, \alpha_i, \beta_i, \ldots, \omega_i, f_1^*, \ldots, f_n^*$ can be complex functions of their arguments.

The system (1, 7) and its canonical form (2, 7) resemble the system

$$dy_i/dx = \Sigma a_{ij} y_j + f_i(x) \quad (i = 1, 2, \ldots, n) \qquad (4, 7)$$

of ordinary differential equations with constant coefficients a_{ij} and its canonical form as given in § 42 (111) of my course on ordinary differential equations (Gostekhizdat, 1949). The difference consists in the fact that, instead of $\partial/\partial x$, the left sides of the corresponding ordinary differential equations have d/dx, and instead of $\partial/\partial y$, the corresponding ordinary differential equations have the multiplier 1 (not written out explicitly). Further, in case of the system of ordinary differential equations (111) the coefficients are constant and the functions f and f^* depend only on a single independent variable, whereas in the case of the partial differential equations considered the coefficients of the derivatives depend on two independent variables and the functions f and f^* depend on these two independent variables as well as on all unknown functions.

The process of reduction of the system (1, 7) to the canonical form (2, 7) is carried out by means of exactly the same change of the unknown functions which is applied in § 43 of my course on ordinary differential equations for a system of linear equations with constant coefficients. All we need worry about now is the proof of the fact that in the neighborhood of the point A the coefficients of the linear substitution described in § 43 are as smooth functions of (x, y) as the coefficients $a_{ij}(x, y)$ of the system (1, 7). To prove this fact we shall have to repeat to some extent § 43.

We shall make use of the method of mathematical induction. For $n = 1$ our theorem which asserts that it is possible to reduce the system (1, 7) to the form (2, 7) by means of a linear transformation with smooth coefficients is obvious. Let us assume that our theorem holds if the number of equations is equal to $n-1$. We wish to show that our theorem also holds for a system of n equations.

Let us multiply the ith equation of the system (1, 7) by k_i, where k_i is an as yet undefined function differentiable in a neighborhood of A. We sum the resulting equations on i and we write the result in the form

$$\frac{\partial\left(\sum_i k_i u_i\right)}{\partial x}=\sum_{i,j}\frac{\partial(a_{ij}k_i u_j)}{\partial y}+\sum_i k_i f_i+\sum_i u_i\frac{\partial k_i}{\partial x}-\sum_{i,j}u_j\frac{\partial(a_{ij}k_i)}{\partial y}.$$

We now define k_i so that the relation

$$\sum_{i,j}a_{ij}k_i u_j\equiv\lambda\sum_i k_i u_i \tag{5, 7}$$

is an identity in u_j. Here λ is some, real or complex, differentiable function of (x,y). Clearly, for (5, 7) to hold it is necessary and sufficient that the coefficients of u_j on both sides of (5, 7) be equal, i.e. that

$$\lambda k_j=\sum_i a_{ij}k_i \quad (j=1,2,\ldots,n) \tag{6, 7}$$

hold. Thus $k_1, k_2, \ldots, k_n$ are defined by means of a system of n linear homogeneous equations in n unknowns. For this system to have a non-trivial solution (which is the only solution of interest to us) it is necessary and sufficient that the determinant of its coefficients vanish. This condition can be written as follows:

$$|\lambda E-\|a_{ij}\||=0. \tag{7, 7}$$

The matrix $\lambda E-\|a_{ij}\|$ is called the *characteristic matrix of the system* (1, 7).

Let λ_i be one of the roots of equation (7, 7). *We assume that all roots of equation* (7, 7) *have the same multiplicity in the neighborhood of A under consideration.* Let the multiplicity of λ_1 in this neighborhood be α_1. Then λ_1 satisfies in this neighborhood an algebraic equation of the form $f^{(\alpha_1-1)}(\lambda_1,x,y)=0$.

Here $f^{(k)}(\lambda,x,y)$ is the kth derivative with respect to λ of the left side of equation (7, 7). Also, throughout this neighborhood

$$f^{(\alpha_1)}(\lambda_1(x,y),x,y)\neq 0.$$

By the implicit function theorem, in some neighborhood of A, $\lambda_1(x,y)$ will be as smooth (i.e. will have as many continuous derivatives) as the coefficients a_{ij}.

We now make the assumption that the matrix

$$\|a_{ij}\|-\lambda_1 E \tag{8, 7}$$

has constant rank r in the neighborhood of A under consideration.†

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† It is easily shown that $r\geqslant n-\alpha_1$. In fact, it is clear that if we put $\lambda=\lambda_1$ then the α_ith derivative of the determinant (7, 7) with respect to λ is a linear combination of the minors of order $(n-\alpha_1)$ of the determinant of (8, 7). Since this derivative is different from zero, one of the minors of order $(n-\alpha_1)$ of the matrix (8, 7) is different from zero.

Then the system (6, 7) has, in this neighborhood of the point A, a solution consisting of functions which do not vanish simultaneously in this neighborhood and which are of the same order of smoothness as the a_{ij}. We denote these functions by k_{1i}. To find such k_{1i}, we make the following observation: if the matrix (8, 7) has rank r in all of the neighborhood of A, then there exist a neighborhood of A and $n-r$ definite equations of the system (6, 7) such that in this neighborhood these $n-r$ equations are consequences of the remaining r equations. Therefore, every system of functions k_{1i} satisfying these r equations in some small neighborhood of A will satisfy all equations of the system (6, 7). Again, in order to find a solution of these r equations (for the sake of brevity we shall refer to these r equations as the equations C_r), we observe that, since the rank of the matrix (8, 7) equals r, it is possible to construct out of the columns of the matrix consisting of the coefficients of the system C_r a quadratic matrix whose determinant does not vanish in some neighborhood of A. We shall regard as unknowns the functions which appear as multipliers of these columns. As for the remaining k_{1i}, we shall put them equal to arbitrary constants not all of which are zero. For the sake of definiteness we shall put them all equal to 1. Then the system C_r will determine uniquely all other k_{1j} as functions of the same order of smoothness as the a_{ij}.

We have thus found in some neighborhood of A the functions k_{1i} $(i = 1, 2, \ldots, n)$ which do not simultaneously vanish in this neighborhood and which are of the same order of smoothness as the a_{ij}. We assume for definiteness that $k_{11} \neq 0$ at A. Clearly, this is no restriction of generality, since this can always be achieved by means of a non-singular linear transformation, specifically, by a permutation of the subscripts of the u_i. We now put

$$z_1 = \sum_{j=1}^{n} k_{1j} u_j.$$

Clearly, the function $z_1(x, y)$ satisfies the equation

$$\frac{\partial z_1}{\partial x} = \lambda_1 \frac{\partial z_1}{\partial y} + f_1^*(x, y, z_1, u_2, \ldots, u_n),$$

where

$$f_1^*(x, y, z_1, u_2, \ldots, u_n) \equiv \sum_i k_i f_i + \sum_i u_i \frac{\partial k_i}{\partial x} - \sum_{i,j} u_i \frac{\partial (a_{ij} k_{1i})}{\partial y} + z_1 \frac{\partial \lambda_1}{\partial y}$$

(cf. formula (5, 7) and the preceding equation).

Now, all considerations in § 43 of my book on ordinary differential equations can be applied without any essential changes. They are considerably simplified if the roots λ of equation (8, 7) are distinct. Then to each root λ_i $(i=1,2,\dots,n)$ of this equation there corresponds a system of functions $k_{ij}(x,y)$ $(j=1,\dots,n)$ which can be determined for each λ_i in the manner used before to determine the functions $k_{1j}(x,y)$ $(j=1,\dots,n)$ for λ_1. The functions $k_{ij}(x,y)$ have as many continuous partial derivatives as the a_{ij}. Also

$$\frac{\partial z_i}{\partial x}=\lambda_i\frac{\partial z_i}{\partial y}+f_i^*(x,y,u_1,\dots,u_n)\quad(i=1,2,\dots,n),$$

where
$$z_i=\sum_{j=1}^{n}k_{ij}u_j\quad(i=1,2,\dots,n).$$

It remains to show that $|k_{ij}|\neq 0$. We give an indirect proof of this fact. Thus let us assume that at some point (x^0,y^0) of the domain of definition of the $k_{ij}(x,y)$, $|k_{ij}(x^0,y^0)|=0$. Then there exist constants C_s not all zero such that

$$\sum_s C_s k_{si}(x^0,y^0)=0\quad(i=1,2,\dots,n).\tag{9,7}$$

Multiplying the ith equation by a_{ij} and summing on i, we get

$$\begin{aligned}\sum_{i,s}C_s k_{si}(x^0,y^0)\,a_{ij}(x^0,y^0)&=\sum_s C_s\sum_i k_{si}(x^0,y^0)\,a_{ij}(x^0,y^0)\\&=\sum_s C_s\lambda_s(x^0,y^0)\,k_{sj}(x^0,y^0).\end{aligned}$$

The last transition was effected using the relation

$$\lambda_s k_{sj}=\sum_i k_{si}a_{ij},$$

which is analogous to (6, 7).

We have thus obtained equalities which are the analogues of (9, 7), $C_s\lambda_s(x^0,y^0)$ taking the place of C_s. In exactly the same manner we obtain

$$\sum_s C_s\lambda_s^m(x^0,y^0)\,k_{si}(x^0,y^0)=0\quad\text{for } m=2,3,\dots,n.$$

Since the determinant of the coefficients of the $C_s k_{si}(x^0,y^0)$ in these equalities (the Vandermonde determinant) is different from zero for distinct $\lambda_1,\dots,\lambda_n$, we get for all s and i

$$C_s k_{si}(x^0,y^0)=0,$$

which is impossible.

Remarks. (1) We required the roots λ of equation (7, 7) to have fixed multiplicity in a neighborhood of A, and we required the matrix (8, 7) to have fixed rank in that same neighborhood for each of the roots of equation (7, 7). These requirements are easily seen to be equivalent to the requirement that the λ-matrix (3, 7) retain the multiplicity of each of its elementary divisors in this neighborhood of A.

(2) If equation (7, 7) has distinct roots in all of the region G, then the preceding considerations prove the existence in some neighborhood of A of a non-vanishing solution $k_{11}, k_{12}, \dots, k_{1n}$ of the system (6, 7) whose functions k_{1i} are of the same order of smoothness as the a_{ij} (cf. footnote, p. 56). It follows that, if all roots of (7, 7) are distinct in all of G, then at every point of G there exists a linear non-singular transformation of the unknown function which reduces the system (1, 7) to the canonical form (2, 7). Also, the coefficients of this transformation are of the same order of smoothness as the a_{ij}, and the system (2, 7) takes the form

$$\frac{\partial v_i}{\partial x} = \lambda_i \frac{\partial v_i}{\partial y} + f_i^*(x, y, v_1, \dots, v_n) \quad (i = 1, 2, \dots, n). \qquad (10, 7)$$

(3) The system (1, 7) is said to be *elliptic* in all of the region G of the (x, y) plane if equation (7, 7) has no real roots at any point of G.

The system (1, 7) is said to be *hyperbolic* in G if there exists at every point (x, y) of G a linear non-singular transformation of the unknown functions u_i with real coefficients $b_{ij}(x, y)$ of the same order of smoothness as the $a_{ij}(x, y)$ which reduces the system (1, 7) to the system (10, 7).

The system (1, 7) is said to be *hyperbolic in the narrow sense* if in the whole region G all roots λ of equation (7, 7) are real and distinct. It is clear that a system hyperbolic in the narrow sense in G is hyperbolic in G.

Similarly, the general linear system of equations with partial derivatives with respect to two independent variables given by

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = \sum_{j=0}^{N} \sum_{k=0}^{n_j - 1} A_{ij}^k(t, x) \frac{\partial^{n_j} u_j}{\partial t^k \partial x^{n_j - k}} + \dots \quad (i = 1, \dots, N) \qquad (11, 7)$$

is said to be *elliptic* in G if the determinant of the matrix

$$\left\| \begin{matrix} \lambda^{n_1} & & \\ & \ddots & \\ & & \lambda^{n_N} \end{matrix} \right\| - \left\| \sum_{k=0}^{n_j - 1} A_{ij}^k \lambda^k \right\|$$

has no real roots at any point of G.

On the other hand, if all roots of this determinant are real and distinct, the system (11, 7) is said to be *hyperbolic in the narrow sense.*

Problem. Show that if the system (11, 7) is hyperbolic in the narrow sense in the region G, then the system of equations of the first order, obtained from (11, 7) in the way (5, 2) was obtained from (3, 2), is hyperbolic in G.

(4) Let us assume that the original system (1, 7) is real, i.e. that all a_{ij}, f_i are real-valued for real values of their arguments. If, in addition, all roots of equation (7, 7) are real, then it is possible to make the system (2, 7) real. For this we need only choose the coefficients of the linear transformation which carries the functions u into functions v as real, and in our case this can be achieved.

On the other hand, if equation (7, 7) has complex roots, then these roots occur in complex conjugate pairs. It is then possible to arrange the system (2, 7) so that to each equation of the system of the form

$$\frac{\partial v_k}{\partial x}=\lambda_k\frac{\partial v_k}{\partial y}+f_k(x,y,v_1,\dots,v_n)$$

there corresponds a complex conjugate equation, i.e. an equation

$$\frac{\partial v_i}{\partial x}=\lambda_i\frac{\partial v_i}{\partial y}+f_i(x,y,v_1,\dots,v_n).$$

with $v_i=\bar{v}_k$, $\lambda_i=\bar{\lambda}_k$, $f_i(x,y,v_1,\dots,v_n)=\overline{f_k(x,y,v_1,\dots,v_n)}$.

Separating real and imaginary parts in these equations and putting

$$v_k=w_k^*+iw_k^{**},$$

$$\lambda_k=a_k+ib_k,$$

$$f_k(x,y,v_1,\dots,v_n)=f_k^*(x,y,w_1,\dots,w_n)+if_k^{**}(x,y,w_1,\dots,w_n),$$

we get

$$\frac{\partial w_k^*}{\partial x}=a_k\frac{\partial w_k^*}{\partial y}-b_k\frac{\partial w_k^{**}}{\partial y}+f_k^*(x,y,w_1,\dots,w_n),$$

$$\frac{\partial w_k^{**}}{\partial x}=b_k\frac{\partial w_k^*}{\partial y}+a_k\frac{\partial w_k^{**}}{\partial y}+f_k^{**}(x,y,w_1,\dots,w_n).$$

The simplest system of this form is the well-known system of Cauchy-Riemann equations:

$$\frac{\partial w_1}{\partial x}=-\frac{\partial w_2}{\partial y},$$

$$\frac{\partial w_2}{\partial x}=\frac{\partial w_1}{\partial y}.$$

Equations of the form

$$\frac{\partial v_k}{\partial x} = \alpha \frac{\partial v_{k-1}}{\partial y} + \lambda \frac{\partial v_k}{\partial y} + f_k(x, y, v_1, \dots, v_n)$$

are likewise decomposed into real and imaginary parts. In this way one can show that it is possible to reduce the system (1, 7) by means of a linear non-singular transformation with real smooth coefficients to a new canonical form in which, in distinction to equations (2, 7), all equations are necessarily real. (Cf. the remark to § 46 of my book on ordinary differential equations.)

CHAPTER II

HYPERBOLIC EQUATIONS

PART I

THE CAUCHY PROBLEM FOR NON-ANALYTIC FUNCTIONS

§ 8. The reasonableness of the Cauchy problem

The Cauchy-Kowalewski theorem asserts the existence of an analytic solution of the Cauchy problem for analytic equations with analytic initial data. Many problems in physics lead to the Cauchy problem for analytic equations with initial data which are not analytic but differentiable a certain number of times. At first sight the following would appear to be a natural method for solving this problem. We approximate the initial functions and their derivatives by means of polynomials. In view of Weierstrass's theorem, the approximating polynomials can be so selected that the difference between them and the corresponding initial functions is arbitrarily small in that part of the plane $t=t_0$ where the Cauchy conditions are prescribed. According to the Kowalewski theorem for analytic equations, it is possible to solve the Cauchy problem if the original initial data are replaced by analytic initial data which approximate the original initial data. It would be natural to expect that the solution of our new problem having the approximating polynomials as initial data would differ by little—at least in the neighborhood of the part of the plane $t=t_0$ where the initial data are prescribed—from the solution of the original problem. However, Hadamard constructed an example which demonstrates the incorrectness of this supposition.

Consider the following Cauchy problem. We are required to find a solution of the Laplace equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}=0, \tag{1, 8}$$

satisfying for $t=0$ the conditions

$$u(0,x)=0, \tag*{(2, 8)$_1$}$$

$$u_t'(0,x)=n^{-k}\sin nx, \tag*{(2, 8)$_2$}$$

where n and k are positive integers. It is easily verified that

$$u(t,x)=n^{-(k+1)}\sin nx\,\frac{\exp\{nt\}-\exp\{-nt\}}{2} \tag{3, 8}$$

is a solution of this problem. Since

$$|u'_t(0,x)|\leqslant n^{-k},$$

it follows that for n sufficiently large the absolute value of $u'_t(0,x)$ will be arbitrarily small for all x. On the other hand, the solution $u(t,x)$ of the Cauchy problem under consideration will, as is seen from (3, 8), assume arbitrarily large values for t arbitrarily small if only n is sufficiently large. The situation will not be changed by the added requirement that together with $|u'_t(0,x)|$ all derivatives of $u'_t(0,x)$ with respect to x of orders up to $k-1$, $k>1$, be small. Incidentally, we do not speak of the smallness of the initial values of the function u itself for these are, by $(2, 8)_1$, identically equal to zero.

Let us assume that we have found a solution of the Cauchy problem for equation (1, 8) with the initial conditions

$$u(0,x)=\phi_0(x),$$
$$u'_t(0,x)=\phi_1(x).$$

Let $u_0(t,x)$ be that solution. Then the function

$$u_0(t,x)+n^{-(k+1)}\sin nx\,\frac{\exp\{nt\}-\exp\{-nt\}}{2}$$

is a solution of the Cauchy problem with the initial data

$$u(0,x)=\phi_0(x),\quad u'_t(0,x)=\phi_1(x)+\frac{1}{n^k}\sin nx.$$

Consequently, a small change of the initial functions and their derivatives of orders up to $k-1$, resulting from the addition of the functions $(2, 8)_1$ and $(2, 8)_2$ to the original initial data, may result in arbitrarily large changes, like those in (3, 8) in the solution of the Cauchy problem, arbitrarily near to the initial value $t=0$.

Usually, Cauchy conditions are found experimentally and, therefore, cannot be determined with absolute precision. However, our preceding considerations show that arbitrarily small errors in the measurements of $u(0,x)$ and $u'_t(0,x)$ and their derivatives with respect to x up to an arbitrary order can entail large errors in the determination of the function $u(t,x)$, satisfying the Laplace equation (1, 8). Therefore, if certain physical considerations were to lead us to the Cauchy problem for the Laplace equation, the solution of this problem would be of no practical value, even if such a solution existed. Fortunately, no physical problems lead to the Cauchy problem for the Laplace equation.

Consider the Cauchy problem in a certain closed region $\bar{G}$ in the space $t, x_1, \ldots, x_n$ for a system of linear equations

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = \sum_{j=1}^{N} \sum_{\substack{k_0+\ldots+k_n \leqslant n_j, \\ k_0 < n_j}} A_{ij}^{(k_0, k_1, \ldots, k_n)}(t, x_1, \ldots, x_n) \frac{\partial^{n_j} u_j}{\partial t^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}} + f_i(t, x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, N). \quad (4, 8)$$

$\bar{G}$ abuts on a region G_0 in the plane $t = t_0$. The Cauchy conditions are prescribed in G_0. This Cauchy problem will be said to be reasonable if certain positive numbers L_1 and L_2 exist such that:

(i) For any functions $\phi_i^{(k)}(x_1, \ldots, x_n)$ given in G_0 and continuous with their derivatives up to order L_1 there exists in the region $\bar{G}$ a unique solution of the system (4, 8) which, for $t = t_0$, satisfies the conditions

$$\frac{\partial^k u_i}{\partial t^k} = \phi_i^{(k)}(x_1, x_2, \ldots, x_n) \quad (k = 0, 1, \ldots, n_i - 1). \quad (5, 8)$$

(ii) For an arbitrary positive ϵ it is possible to find an $\eta > 0$ such that, if the functions $\phi_i^{(k)}$ and all of their derivatives with respect to $x_1, x_2, \ldots, x_n$ of order up to L_2 change by less than η in G_0, then the solution of the Cauchy problem changes by less than ϵ in all of $\bar{G}$.

It follows from the preceding considerations that only those solutions of the Cauchy problem for which this problem is reasonable are of interest in physics (the word 'physics' is always used in the most inclusive sense). On the other hand, Hadamard's example shows that the Cauchy problem is not necessarily reasonable for all equations.

Our considerations concerning the reasonableness of the Cauchy problem indicate that other boundary-value problems involving partial derivatives are of interest in the natural sciences only if their solutions depend continuously, in some sense, on the initial data; in other words, if they are reasonable.† For every type of equation there exist boundary-value problems which are 'reasonable' for this particular type of equation.

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† In each particular case the notion of reasonableness must be precisely defined.

When determining the reasonableness of the Cauchy problem for non-linear equations, it is essential that we consider as possible initial functions $\phi_i^{(k)}(x_1, \ldots, x_n)$ only functions which are close to certain definite functions $\tilde{\phi}_i^{(k)}(x_1, \ldots, x_n)$. It can happen that the Cauchy problem is reasonable close to one system of functions $\tilde{\phi}_i^{(k)}(x_1, \ldots, x_n)$ and is not reasonable close to another system of functions $\tilde{\phi}_i^{(k)}(x_1, \ldots, x_n)$.

In almost all cases considered thus far the formulation of such problems has been suggested by physical considerations. In particular, all of the problems given in § 1 are reasonable.

In this chapter we shall prove the reasonableness of the Cauchy problem for: (*a*) the 'wave equation' of the second order with an arbitrary number of independent variables (see para. 1, § 1) for a suitable inclination of the hyperplane on which the initial conditions are prescribed; (*b*) a linear hyperbolic system of partial differential equations of the first order in two independent variables. If we now take into account the result contained in the problem following remark (3) in § 7, we may conclude that the Cauchy problem is reasonable for linear systems of the form (11, 7) in two independent variables which are hyperbolic in the narrow sense.

§ 9. The notion of generalized solutions

In the preceding section we discussed the reasonableness of the Cauchy problem for sufficiently smooth initial conditions. It is not usual, however, for physical problems to lead to initial conditions which are sufficiently smooth to justify the claim that the problem in question has a solution. It often happens that a boundary-value problem has no differentiable solution if the initial conditions are not continuous and differentiable a sufficient number of times. In that case the application of the concept of 'generalized solutions' of differential equations turns out to be extremely useful.

S. L. Sobolev was the first to introduce the concept of generalized solutions of differential equations and to apply it in a systematic manner. Following him, *we call a system of functions* $u_1, u_2, \ldots, u_N$ *a generalized solution of a system of differential equations in a region* G, *if there exists an infinite sequence of solutions* $(u_1^{(k)}, \ldots, u_N^{(k)})$ *of this system which converges uniformly to* $(u_1, \ldots, u_N)$, *i.e. if*

$$\sup_{\substack{P \in G \\ k \to \infty}} \sum_{i=1}^{N} | u_i(P) - u_i^{(k)}(P) | \to 0. \qquad (1, 9)$$

Example. We consider the equation

$$\partial^2 u / \partial x \, \partial y = 0.$$

Every function $u(x, y)$ of the form

$$u = \phi(x) + \psi(y)$$

will be an ordinary (as against generalized) solution of this equation provided that the functions $\phi(x)$ and $\psi(y)$ are twice continuously differentiable. We know that it is possible to represent arbitrary functions $\phi(x)$ and $\psi(y)$, which are continuous (but not necessarily differentiable) in $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$, respectively, as uniform limits of sequences of twice continuously differentiable functions $\phi^k(x)$ and $\psi^k(y)$, respectively. Therefore, for any continuous functions $\phi(x)$ and $\psi(y)$, the sum

$$\phi(x) + \psi(y)$$

represents a generalized solution of equation (1, 9) in the sense indicated above.

S. L. Sobolev has proved the existence and uniqueness of generalized solutions for the most typical boundary-value problems for partial differential equations. In this connection a special definition of the meaning of boundary conditions for the generalized solutions is required.

The introduction of generalized solutions by the method indicated above does not extend the class of ordinary solutions of either elliptic equations (this will be shown later; cf. Harnack's theorem) or parabolic equations. In the case of hyperbolic equations, however, the resulting extension is of fundamental importance.

In order to ensure the existence of ordinary solutions in the case of the fundamental boundary-value problems, it is necessary to impose rigid smoothness conditions on the functions prescribed on the boundary. What makes the concept of generalized solutions convenient is the fact that, in order to ensure their existence, we need not impose equally rigid conditions of smoothness on the functions prescribed on the boundary. To illustrate the point we solve the Cauchy problem for the equation

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2 \tag{2, 9}$$

with initial conditions

$$u(0, x) = \phi(x), \quad u'_t(0, x) = \psi(x). \tag{3, 9}$$

It can easily be checked that the solution of this problem is given by the function

$$u(t, x) = \frac{\phi(x+t) + \phi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\, d\xi. \tag{4, 9}$$

This formula gives the ordinary solution of equation (2, 9) only under the assumption that $\phi(x)$ has continuous derivatives up to and including the second order and that $\psi(x)$ has continuous derivatives of the first order. At the same time, for arbitrary continuous $\phi(x)$, $\phi'(x)$ and $\psi(x)$, formula (4, 9) gives a generalized solution of equation (2, 9) satisfying the conditions (3, 9).

Consideration of the generalized solutions of equation (2, 4) is all the more essential because, usually, the values of the functions $\phi(x)$ and $\psi(x)$ are known only approximately. Likewise, the corresponding function $u(t,x)$ given by formula (4, 9) represents only an approximation to the exact solution of the problem. Whether this approximation is an ordinary or generalized solution of equation (2, 9) does not matter. What matters is that this approximation differs little from the true solution if the functions $\phi(x)$ and $\psi(x)$ differ from the true initial conditions $u(0,x)$ and $u'_t(0,x)$ by a uniformly small amount.

Remark. S. L. Sobolev also applies the name generalized solution of a differential equation in a region G to any function u which is the limit, in the sense of mean convergence, of a sequence of ordinary solutions of this equation; i.e. if $u^{(k)}$ is an ordinary solution, then a generalized solution is a function u for which

$$\int_G [u-u^{(k)}]^2 \, d\sigma \to 0 \quad \text{as} \quad k \to \infty.$$

Generalized solutions defined in this manner can be discontinuous. For details see S. L. Sobolev, *Uravnenia matematicheskoi fiziki* [*Equations of Mathematical Physics*], Gostekhizdat (1947), in particular, pp. 292, 298 and 301.

§ 10. The Cauchy problem for hyperbolic systems in two independent variables.

1. We consider the system

$$\partial u_i/\partial t - \lambda_i(\partial u_i/\partial x) = \sum_{j=1}^{N} a_{ij}(t,x)\, u_j + b_i(t,x). \qquad (1, 10)\dagger$$

---

† Except for minor modifications all considerations in this section are applicable to systems of the form

$$\frac{\partial u_i}{\partial t} - \lambda_i \frac{\partial u_i}{\partial x} = f_i(t, x, u_1, \ldots, u_N) \quad (i = 1, 2, \ldots, N)$$

under the assumption that the functions $f_i(t, x, u_1, \ldots, u_N)$ have continuous derivatives up to and including the second order (cf. the proof of existence of a solution of the equation

$$dy/dx = f(x, y)$$

by means of the method of successive approximations).

We assume that the system is hyperbolic in all of the region G, i.e. that all λ_i are real functions of (t, x). It may be assumed without loss of generality that

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N.$$

Through each point of our region pass N real characteristics L_i whose slopes (with respect to the x-axis) are given by $k_i = -\lambda_i^{-1}$ (see example 5, § 3).

If it is not assumed that the coefficients of the system (1, 10) are analytic, then it is not possible to use the Cauchy-Kowalewski theorem to support claims of solvability of the Cauchy problem for this system. We assume that the functions a_{ij}, b_i and λ_i are continuous and have continuous partial derivatives in some closed region $\bar{G}$ bounded by the interval $[a, b]$ of the x-axis and by the characteristics L_1 and L_N at the points a and b, respectively (see fig. 5).† We prescribe on the interval $[a, b]$ N continuously differentiable functions $\phi_1(x), \ldots, \phi_N(x)$, and we formulate the Cauchy problem for the system (1, 10) as follows:

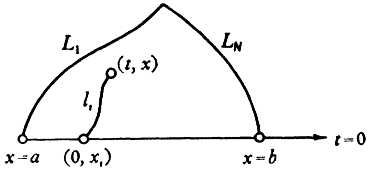

Figure 5

Find a solution $u_1, u_2, \ldots, u_N$ *of the system* (1, 10) *which is continuous in* G, *has continuous first derivatives, and is such that for* $t = 0$

$$u_i(x, 0) = \phi_i(x). \tag{2, 10}$$

Under these assumptions the problem has a solution and the solution is unique.

Proof. We consider the ith equation of the system (1, 10). Its left side is equal, to within a multiplicative factor, to the derivative of the function $u_i(t, x)$ in the direction of the curve L_i. In fact, if we denote by α_i the angle at the point (t, x) between the curve L_i and the x-axis, then, as already indicated,

$$\tan \alpha_i = -(\lambda_i)^{-1}.$$

Consequently,

$$\cos \alpha_i = -\lambda_i/(1+\lambda_i^2)^{\frac{1}{2}}, \quad \sin \alpha_i = 1/(1+\lambda_i^2)^{\frac{1}{2}}$$

---

† The curves L_1 and L_N need not intersect. Consequently, the region G can be infinite.

For what follows it is essential that G be bounded. This can always be achieved by bounding G, if necessary, by means of a straight line $t = T$.

All our further considerations are applicable if G is contained in the half-plane $t < 0$.

and
$$\frac{\partial u_i}{\partial s_i}=\left(\frac{\partial u_i}{\partial t}-\lambda_i\frac{\partial u_i}{\partial x}\right)\frac{1}{(1+\lambda_i^2)^{\frac{1}{2}}}.$$

Here s_i denotes arc length along the ith characteristic, and $\partial/\partial s_i$ denotes differentiation in the direction of the ith characteristic.

The system (1, 10) can be rewritten as follows:

$$(1+\lambda_i^2)^{\frac{1}{2}}(\partial u_i/\partial s_i)=\sum_{j=1}^{N}a_{ij}u_j+b_i \quad (i=1,\dots,N). \tag{3, 10}$$

If we denote by $\underset{i}{du_i}$ the differential of the function u_i along the curve L_i, then, using (3, 10), we have

$$\underset{i}{du_i}=(\Sigma a_{ij}u_j+b_i)\,ds_i/(1+\lambda_i^2)^{\frac{1}{2}},$$

and since
$$ds_i=(1+\lambda_i^2)^{\frac{1}{2}}\,dt,$$
it follows that
$$\underset{i}{du_i}=(\Sigma a_{ij}u_j+b_i)\,dt \quad (i=1,2,\dots,N). \tag{4, 10}$$

We now fix an arbitrary point (t,x) of the region $\bar{G}$, and we denote by l_i the part of the curve L_i, extending from the point (t,x) to the point of intersection $(0,x)$ of L_i with the interval $[a,b]$ on the x-axis (see fig. 5). We now integrate the ith relation in formula (4, 10) along the arc l_i from $(0,x)$ to (t,x). We thus obtain a system of integral equations

$$u_i(t,x)-u_i(0,x_i)=\int_{l_i}\left(\sum_{j=1}^{N}a_{ij}u_j+b_i\right)dt \quad (i=1,2,\dots,N),$$

or, in view of the initial conditions (2, 10), the system

$$u_i(t,x)=\phi_i(x_i)+\int_{l_i}\left(\sum_{j=1}^{N}a_{ij}u_j+b_i\right)dt. \tag{5, 10}$$

It is clear that every solution of the system (1, 10) satisfying the initial conditions (2, 10) is a solution of the system (5, 10). Conversely, if we have a solution of the system of integral equations (5, 10), and if the functions which make up this solution have continuous derivatives with respect to x and y, then, reversing the order of the steps which led us from (1, 10) to (5, 10), we can convince ourselves that the solution of the system (5, 10) is also a solution of the Cauchy problem as formulated. This problem has thus been reduced to that of proving the existence of a continuously differentiable solution of the system (5, 10).

We construct a system of successive approximations to the solution of (5, 10) in the following manner: we put

$$u_i^0(t,x)=\phi_i(x_i) \quad (i=1,\ldots,N)$$

$$u_i^{(1)}(t,x)=\phi_i(x_i)+\int_{l_i}(\Sigma a_{ij}(t,x)\,u_j^{(0)}+b_i(t,x))\,dt \quad (i=1,2,\ldots,N)$$

and, in general,

$$u_i^{(n+1)}(t,x)=\phi_i(x_i)+\int_{l_i}(\Sigma a_{ij}(t,x)\,u_j^{(n)}+b_i(t,x))\,dt \quad (i=1,\ldots,N).$$

The latter equality should rather be written as

$$u_i^{(n+1)}(t,x)=\phi_i[x_i(0,t,x)]$$
$$+\int_0^t\left[\sum_j a_{ij}(\tau,x_i(\tau,t,x))\,u_j^{(n)}(\tau,x_i(\tau,t,x))+b_i(\tau,x_i(\tau,t,x))\right]d\tau.$$

Here $x=x_i(t,t^0,x^0)$ is the equation of the ith characteristic through the point (t^0,x^0). If we can demonstrate the uniform convergence of the sequence $u_i^{(n)}(t,x)$ in the closed region $\bar{G}$, then the system of limit functions $u_i(t,x)$ will satisfy equations (5, 10). The uniform convergence of the sequence $u_i^{(n)}(t,x)$ is equivalent to the uniform convergence of the series

$$u_i^{(0)}(t,x)+\sum_{n=0}^{\infty}[u_i^{(n+1)}(t,x)-u_i^{(n)}(t,x)]. \tag{6, 10}$$

To prove the uniform convergence of this series we shall construct for this series a majorizing series of numbers. Since the functions $u_i^{(0)}(x,t)$ and $u_i^{(1)}(x,t)$ are continuous in the closed region $\bar{G}$, they are bounded in $\bar{G}$.

We put $M=\max\{|u_1^{(0)}|,\ldots,|u_N^{(0)}|,|u_1^{(1)}|,\ldots,|u_N^{(1)}|\}$ in the region $\bar{G}$. Then

$$\left.\begin{aligned}|u_i^{(0)}(t,x)|&\leqslant M,\\ |u_i^{(1)}-u_i^{(0)}|&\leqslant 2M,\end{aligned}\right.\quad (t,x)\in\bar{G}.$$

We denote A by $\max|a_{ij}|$ in $\bar{G}$ for all $i,j=1,\ldots,N$. Then

$$|u_i^{(2)}(t,x)-u_i^{(1)}(t,x)|\leqslant\int_{l_i}\sum_{j=1}^{N}|a_{ij}|\,.\,|u_j^{(1)}-u_j^{(0)}|\,dt\leqslant 2MANt,$$

$$|u_i^{(3)}(t,x)-u_i^{(2)}(t,x)|\leqslant\int_{l_i}\sum_{j=1}^{N}|a_{ij}|\,.\,|u_j^{(2)}-u_j^{(1)}|\,dt\leqslant 2MA^2N^2\frac{t^2}{2}.$$

We now assume that

$$|u_i^{(n)}(t,x)-u_i^{(n-1)}(t,x)|\leqslant 2M\frac{A^{n-1}N^{n-1}t^{n-1}}{(n-1)!}.$$

Then

$$| u_i^{(n+1)}(t,x) - u_i^{(n)}(t,x) | \leqslant \int_{l_i} \sum_{j=1}^{N} | a_{ij} | \cdot | u_j^{(n)} - u_j^{(n-1)} | \, dt \leqslant 2M \frac{A^n N^n t^n}{n!}.$$

Thus, for an arbitrary n,

$$| u_i^{(n+1)}(t,x) - u_i^{(n)}(t,x) | \leqslant 2M \frac{(ANt)^n}{n!}.$$

The region G is bounded, however. Therefore, if T is chosen to be greater than the values of t in G, we have in all of G

$$| u_i^{(n+1)} - u_i^{(n)} | \leqslant 2M \frac{(ANT)^n}{n!}.$$

Since the series $\sum \frac{(ANT)^n}{n!}$ converges, it follows that the series (6, 10) converges uniformly in the whole closed region $\bar{G}$. This proves the existence and continuity of the solution of the system (5, 10).

We prove in an analogous manner the uniqueness of the continuous and, consequently, bounded solution of (5, 10) in $\bar{G}$. We assume that we have two such solutions of (5, 10) denoted by $u_1, \ldots, u_N$ and $\tilde{u}_1, \ldots, \tilde{u}_N$, respectively. Substituting both solutions in the system and subtracting corresponding equations, we get

$$u_i(t,x) - \tilde{u}_i(t,x) = \int_{l_i} \sum_{j=1}^{N} a_{ij}(u_j - \tilde{u}_j) \, dt.$$

We now assume that

$$\max_{i=1,\ldots,N} | u_i - \tilde{u}_i | = M > 0.$$

Then, making the estimates which we made before in the existence proof, we get that

$$M \leqslant M \frac{(ANT)^n}{n!}$$

for an arbitrary n. Taking n sufficiently large we arrive at a contradiction. This proves that $M = 0$ and

$$u_i(t,x) = \tilde{u}_i(t,x);$$

i.e. the solution is unique.

To complete our proof we must still show that the solution functions $u_i(t,x)$ have continuous first derivatives with respect to x and y. To prove this it is enough to show that the $u_i(t,x)$ have at all points continuous first derivatives in the direction of l_i and with respect to x, for this fact and the smoothness of the l_i prove the continuity of the derivatives with respect to x and y in all of the region G.

The existence and continuity of the derivatives of the $u_i(t,x)$ in the direction of l_i follows directly from the system (5, 10) and from the continuity of the solution obtained. To prove the existence and continuity of the derivatives $\partial u_i/\partial x$ we observe, first of all, that the assumed continuous differentiability of $\phi_i(x)$, $\lambda_i(t,x)$, $a_{ij}(t,x)$, $b_i(t,x)$ implies that all approximations constructed in proving the existence of a solution have continuous derivatives with respect to x. We now differentiate with respect to x the equality which defines the $(n+1)$st approximation. We get†

$$\begin{aligned}\frac{\partial u_i^{(n+1)}(t,x)}{\partial x} &= \phi_i'[x_i(0,t,x)]\frac{\partial x_i(0,t,x)}{\partial x}\\ &+\int_0^t\Bigg[\sum_j \frac{\partial a_{ij}(\tau,x_i(\tau,t,x))}{\partial x}u_j^{(n)}(\tau,x_i(\tau,t,x))\\ &+\sum_j a_{ij}(\tau,x_i(\tau,t,x))\frac{\partial u_j^{(n)}(\tau,x_i(\tau,t,x))}{\partial x_i}\cdot\frac{\partial x_i(\tau,t,x)}{\partial x}\\ &+\frac{\partial b_i(\tau,x_i(\tau,t,x))}{\partial x}\Bigg]d\tau \quad (i=1,2,\ldots,N).\end{aligned}$$

In view of the assumption made about the system (1, 10), we can prove the uniform convergence of the sequence $\partial u_i^{(n)}/\partial x$ $(n=1,2,\ldots)$, using exactly the same method which we used to prove the convergence of the sequence $u_i^{(n)}$. It follows that

$$\lim_{n\to\infty}\frac{\partial u_i^{(n)}}{\partial x}=\frac{\partial u_i}{\partial x},$$

and that this function is continuous, which is what we set out to prove.

If the functions $\phi_i(x)$ were continuous without being differentiable, the construction described in the beginning of this section would give us only a generalized solution of the system (1, 10) (cf. para. 2 below).

2. We have shown the existence and uniqueness of the solution of the Cauchy problem for the system (1, 10) in the class of functions possessing continuous derivatives of the first order. To demonstrate the reasonableness of this problem we prove the following theorem (cf. § 8):

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† In view of the assumed continuity of the derivatives of λ_i, the coordinate x_i of the point of intersection of l_i with the straight line $t=\tau$ is a function which is continuously differentiable with respect to x and t. The limits of integration with respect to t in the line integral do not vary when x varies.

If the initial functions $\phi_i(x)$ *of the Cauchy problem are replaced by functions* $\psi_i(x)$ *which differ from the* $\phi_i(x)$ *by less than* η, *then the functions* $v_i(t,x)$, *which constitute the solution of the Cauchy problem with the* $\psi_i(x)$ *as initial functions, differ from the corresponding functions* $u_i(t,x)$ *by less than* ϵ, *and* $\epsilon \to 0$ *if* $\eta \to 0$.

We put

$$\left.\begin{aligned} \phi_i(x) - \psi_i(x) &= \eta_i(x), \\ u_i(t,x) - v_i(t,x) &= z_i(t,x). \end{aligned}\right\} \qquad (7, 10)$$

The functions $z_i(t,x)$ satisfy the integral equations

$$z_i(t,x) = \eta_i(x_i) + \int_{l_i} (\Sigma a_{ij} z_j)\, dt. \qquad (8, 10)$$

Now let us put

$$\max_{(t,\,x)\epsilon G} |z_i(t,x)| = \epsilon.$$

Then, by means of an estimate analogous to that used in the existence proof, we obtain

$$\epsilon \leqslant \eta + A\epsilon N t. \qquad (9, 10)$$

Replacing each z_i in the integral equation (8, 10) by the right side of the inequality (9, 10) and repeating our considerations, we obtain

$$\epsilon \leqslant \eta(1 + ANt) + \epsilon \frac{A^2N^2t^2}{2}.$$

Repeating this operation n times, we infer the correctness of the inequality

$$\epsilon \leqslant \eta\left(1 + ANt + \ldots + \frac{(ANt)^{n-1}}{(n-1)!}\right) + \epsilon \frac{(ANt)^n}{n!}.$$

Going over to the limit as $n \to \infty$, we get

$$\epsilon \leqslant \eta \exp\{ANt\}.$$

From this it is clear that if t is bounded, $\epsilon \to 0$ as $\eta \to 0$.

3. We conclude this section with a brief description of a method (method of finite differences) which enables us to obtain an approximate solution of the Cauchy problem formulated under para. 2.

Let the initial functions be prescribed on the segment $[a,b]$ of the x-axis. To approximate the values of the solution functions $u_i(t,x)$ of the system (1, 10), which for $t = 0$ are equal to the given functions $\phi_i(x)$, we proceed as follows.

We fix an integer n and we divide the interval $[a,b]$ into n equal parts of length $h = (b-a)/n$ each. We then draw the straight lines $x = a + ph$ and $t = qh$ for integral values of p and q which are chosen in such a manner that the region G of existence of the solution of the Cauchy problem (cf. para. 1 of this section) is covered with a net of squares of side length h. We number each of the vertices of our squares by means of two indices. Specifically, M_{pq} is to be the vertex which is the point of intersection of the straight lines $x=a+ph$ and $t=qh$. We are given the values of the unknown functions $u_i(t,x)$ at all points M_{p0}, namely,

$$u_i(0,a+ph)=\phi_i(a+ph)=\phi_i(M_{p0}).$$

We now describe a procedure by means of which it is possible to find approximately the values of $u_i(t,x)$ at all those vertices of the network of squares which lie in G. At each point M_{p0} there are given the coefficients of the system (1, 10) and N numbers

$$\lambda_i(M_{p0})=\lambda_i^{p0} \quad (i=1,\ldots,N).$$

At each point M_{p0} we draw straight-line segments with slopes $k_i^{p0}=-(\lambda_i^{p0})^{-1}$ of such length that they intersect the straight line $t=h$. We now find the values of $u_i(t,x)$ at the other end-points of these segments. To this end we make use of the form (4, 10) of the system (1, 10), and we replace the differential along the ith characteristic by the increment, and the equality sign by the sign of approximate equality. We thus obtain the relation

$$\underset{i}{\Delta} u_i \approx (\Sigma a_{ij}u_j+b_i)\,h,$$

which allows us to find the increment $\underset{i}{\Delta} u_i$ associated with going from the point M_{p0} along the ith characteristic (more accurately, along the tangent to this characteristic) to the line

$$t=h.$$

Adding the increments thus obtained to the values of the functions at the points M_{p0}, we find the values of each of the functions u_i at points of the line $t=h$. The values of different functions will, in general, be determined at different points. Applying some kind of interpolation procedure to the values of u_i as determined at points of the line $t=h$, we now determine its values at the points M_{p1}, the vertices of the network lying on that line. Repeating this procedure, we can now determine the values of $u_i(t,x)$ at points of G on the line $t=2h$. In this way, using interpolation and determination of the values of $u_i(t,x)$ a sufficient number of times, we can find approximate values of all the functions $u_i(t,x)$ at all vertices in G of our network.

It can be shown that, as $n \to 0$, the approximate values of the functions converge uniformly to a limit which gives the exact solution of the Cauchy problem. Consequently, if n is sufficiently large, the approximate values found by the above method differ by an arbitrarily small quantity from the true solution.

If $N = 2$, the procedure for computing an approximate solution of the Cauchy problem is considerably simplified. In this case there are only two families of characteristics. We divide the segment $[a, b]$ of the x-axis on which are prescribed the initial values of u_1 and u_2 into small intervals. Next we draw at the points of division tangents to the characteristics. Using the method outlined above, we now determine approximate values of u_1 and u_2 at those points of intersection of the tangents to the characteristics which are closest to the interval $[a, b]$. Beginning with these points of intersection, we again draw tangents to the characteristic and again determine approximate values of u_1 and u_2 at intersection points closest to $[a, b]$, etc. In this way we obtain the values of u_1 and u_2 on a sufficiently dense point set provided that the initial subdivision of the segment $[a, b]$ is sufficiently fine. As can be seen, we can, in this case, dispense both with the network of squares and with the interpolation procedure.

§ 11. The Cauchy problem for the wave equation. Uniqueness of the solution

Let the function $u(t, x_1, x_2)$ satisfy the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \tag{1,11}$$

for points in the interior of a circular cone K whose axis is parallel to the t-axis, whose vertex is at a point A, and whose generators form an angle $\alpha = 45°$ with the t-axis. Further, let the function $u(t, x_1, x_2)$ and all its derivatives of order up to the second be continuous in the interior and on the boundary of K.

Then the value of $u(t, x_1, x_2)$ at the point A is uniquely determined by the values of u and of $\partial u/\partial n$ at the base of the cone in the plane $t = t_0$.

The cone K is said to be *characteristic*. It can be easily seen that the surface of K (without its base) is a characteristic surface in the sense of para. 1, § 3.

The theorem is true regardless of whether the t coordinate of the point A is greater than t_0 or less than t_0.

Remarks. (1) Let us replace the cone whose generators form a 45° angle with the t-axis by another cone whose generators form an angle $\alpha = \arctan a^{-1}$ ($a = \text{const.}$) with the t-axis. Then, in formulating our theorem, we could replace equation (1, 11) by the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right). \tag{2, 11}$$

In fact, equation (2, 11) can be reduced to equation (1, 11) by replacing at by t.

(2) We can always assume that $t_0 = 0$. The case of an arbitrary t_0 reduces to the latter case if in place of the independent variable t we introduce a new independent variable $t^* = t - t_0$. This transformation does not change the form of equation (1, 11).

(3) Let us assume that we are given in the plane $t = 0$ a region G_0 with boundary L. We construct cones K with bases in the region G_0, with axes parallel to the t-axis, and with generators forming a ± 45° angle with the t-axis. It then follows from our theorem that prescribing the values of u and $\partial u/\partial t$ in the region G_0 determines uniquely the solution of equation (1, 11) in the region G of the space (t, x_1, x_2), which is filled out by cones K. Thus, for instance, prescribing u and $\partial u/\partial t$ in the square $|x_1| < a$, $|x_2| < a$ determines uniquely a twice continuously differentiable solution $u\ (t, x_1, x_2)$ of equation (1, 11) in the interior of two pyramids with this square as common base and with sides forming an angle of 45° with this common base. Prescribing the values of u and $\partial u/\partial t$ on the base of a cone K determines u in the interior and on the boundary of K.

(4) Prescribing the values of u and $\partial u/\partial t$ on some circle G_0 in the plane (x_1, x_2) does not determine the solution $u(t, x_1, x_2)$ of equation (1, 11) at any point B which lies outside the totality of cones with bases in the circle G_0, with axes parallel to the t-axis and with generators forming a 45° angle with the t-axis. To prove this it suffices to show that there exists a solution $\tilde{u}(t, x_1, x_2)$ for which $\tilde{u}$ and $\partial\tilde{u}/\partial t$ are equal to 0 in G_0 and for which $\tilde{u}(B) \neq 0$. To construct such a solution we observe that, if $f(z)$ is any twice continuously differentiable function and if $\alpha_1^2 + \alpha_2^2 = 1$, then the function

$$f(t + \alpha_1 x_1 + \alpha_2 x_2) \tag{3, 11}$$

is a solution of equation (1, 11).

The function (3, 11) is constant at all points of any of the planes

$$t + \alpha_1 x_1 + \alpha_2 x_2 = C, \tag{4, 11}$$

each of which forms a 45° angle with the t-axis. Let us choose α_1 and α_2 so that the member of the family (4, 11) which goes through the point B does not intersect the circle G_0. It is now possible to choose a twice continuously differentiable function $f(z)$ such that $f(t+\alpha_1 x_1+\alpha_2 x_2)$ is different from zero at the point B and such that f and its first derivative vanish in G_0. Then

$$\tilde{u}(t, x_1, x_2) = f(t+\alpha_1 x_1+\alpha_2 x_2)$$

will be the required solution.

(5) The uniqueness proof given below applies to a twice continuously differentiable solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2}$$

for any value of n. It would then be necessary to replace the three-dimensional cone mentioned in the formulation of the theorem by a cone in the space of $n+1$ dimensions with axis parallel to the t-axis and with generators forming a 45° angle with the t-axis. For $n=1$ this cone reduces to a triangle with base parallel to the x-axis and with sides forming with this axis an angle of 45°.

Proof of the uniqueness theorem. Let us assume that there exist in the interior of K and on its surface two twice continuously differentiable solutions $u_1(t, x_1, x_2)$, $u_2(t, x_1, x_2)$ of (1, 11). Let the values of u_1 and u_2 and the values of u'_{1t} and u'_{2t} coincide on the base K. Then the difference

$$u(t, x_1, x_2) = u_2(t, x_1, x_2) - u_1(t, x_1, x_2)$$

must satisfy the homogeneous equation (1, 11) in the interior of K, and $u(t, x_1, x_2)$ and $u'_t(t, x_1, x_2)$ must vanish on the surface of K. We shall have proved our uniqueness theorem if we can show that $u(t, x_1, x_2) = 0$ at the vertex of K.

To prove this we integrate the expression

$$\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}\right)$$

over the interior of K. This expression vanishes at all points of the interior of K, for the function u satisfies equation (1, 11) at those points. Since

$$\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2,$$

and

$$\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_i}\right) - \frac{\partial^2 u}{\partial t\,\partial x_i}\frac{\partial u}{\partial x_i},$$

it follows that

$$0=\iiint_K \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}\right) dt\,dx_1\,dx_2$$

$$=\iiint_K \left\{\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]\right.$$

$$\left.-\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_2}\right)\right\} dt\,dx_1\,dx_2.$$

We now transform this triple integral into a double integral and in so doing make use of Green's theorem. If K_1 denotes the surface of the cone K exclusive of its base and if C denotes its base, then all we need consider is the integral over K_1; for, by our initial conditions,

$$\partial u/\partial t=\partial u/\partial x_1=\partial u/\partial x_2=0$$

at all points of C. We thus have

$$0=\frac{1}{2}\iint_{K_1}\left\{\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]\cos(n,t)\right.$$

$$\left.-2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_1}\cos(n,x_1)-2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_2}\cos(n,x_2)\right\}d\sigma. \qquad (5, 11)\dagger$$

But on the surface of the characteristic cone (exclusive of its base)

$$\cos^2(n,t)-\cos^2(n,x_1)-\cos^2(n,x_2)=0. \qquad (6, 11)$$

Multiplying and dividing the integrand by $\cos(n,t)$ and using the relation (6, 11), we get from (5, 11)

$$\frac{1}{2\cos(n,t)}\iint_{K_1}\left\{\left(\frac{\partial u}{\partial t}\cos(n,x_1)-\frac{\partial u}{\partial x_1}\cos(n,t)\right)^2\right.$$

$$\left.+\left(\frac{\partial u}{\partial t}\cos(n,x_2)-\frac{\partial u}{\partial x_2}\cos(n,t)\right)^2\right\}d\sigma=0. \qquad (7, 11)$$

Here $\cos(n,t)$ has been taken out from under the integral sign, for this quantity is constant on K_1 (note that $\cos(n,t)=(2)^{-\frac{1}{2}}$ for $t>t_0$ and $\cos(n,t)=-(2)^{-\frac{1}{2}}$ for $t<t_0$).

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† We call the reader's attention to the fact that in transforming the integral

$$\iiint_K \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}\right) dt\,dx_1\,dx_2,$$

we have made use of the continuity of the first derivatives of u inside and on the boundary of K and of the integrability over K of the second derivatives of u. The second derivatives of u are always integrable over K if they are continuous on K and on its boundary.

The equality (7, 11) implies that on the surface of the cone K (exclusive of its base)

$$\frac{u_t'}{\cos(n,t)}=\frac{u_{x_1}'}{\cos(n,x_1)}=\frac{u_{x_2}'}{\cos(n,x_2)}=v. \tag{8, 11}$$

If we denote by m the direction of any of the generators of K, then, using (8, 11), we get

$$\begin{aligned}\frac{\partial u}{\partial m}&=u_t'\cos(m,t)+u_{x_1}'\cos(m,x_1)+u_{x_2}'\cos(m,x_2)\\&=v[\cos(n,t)\cos(m,t)+\cos(n,x_1)\cos(m,x_1)\\&\qquad+\cos(n,x_2)\cos(m,x_2)]\\&=v\cos(m,n)=0.\end{aligned}$$

Thus, on the surface of the cone K the derivative of u in the direction of a generator vanishes. From this it follows that the function u vanishes at the vertex of the cone, for it vanishes at the points of its base. This concludes our proof of uniqueness.

§ 12. Formulas giving the solution of the Cauchy problem for the wave equation

1. Let G_0 in the space (x_1, x_2, x_3) be the domain of definition of a three times continuously differentiable function $\phi_0(x_1, x_2, x_3)$ and a twice continuously differentiable function $\phi_1(x_1, x_2, x_3)$. We wish to determine in G (the region described in remark (3) in § 11) the solution $u(t, x_1, x_2, x_3)$ of the equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}, \tag{1, 12}$$

satisfying for $t=0$ the conditions

$$u(0, x_1, x_2, x_3)=\phi_0(x_1, x_2, x_3), \qquad (2, 12)_1$$

$$u_t'(0, x_1, x_2, x_3)=\phi_1(x_1, x_2, x_3). \qquad (2, 12)_2$$

We first seek a solution $u_\phi(t, x_1, x_2, x_3)$ of equation (1, 12) when the initial conditions have the special form

$$u_\phi(0, x_1, x_2, x_3)=0, \qquad (3, 12)_1$$

$$u_{\phi t}'(0, x_1, x_2, x_3)=\phi(x_1, x_2, x_3). \qquad (3, 12)_2$$

It is then easily checked that the function

$$v(t, x_1, x_2, x_3)=\partial u_\phi/\partial t$$

satisfies for $t=0$ the conditions

$$v(0, x_1, x_2, x_3) = \phi(x_1, x_2, x_3),$$
$$v'_t(0, x_1, x_2, x_3) = \frac{\partial^2 u_\phi}{\partial t^2} = \frac{\partial^2 u_\phi}{\partial x_1^2} + \frac{\partial^2 u_\phi}{\partial x_2^2} + \frac{\partial^2 u_\phi}{\partial x_3^2} = 0.$$

Therefore, the solution of equation (1, 12) satisfying both conditions (2, 12) is given by the formula

$$u = \partial u_{\phi_0}/\partial t + u_{\phi_1}. \tag{4, 12}$$

Thus, the general Cauchy problem for equation (1, 12) is reduced to the problem of finding u_ϕ. We claim that

$$u_\phi(t, x_1, x_2, x_3) = \frac{1}{4\pi} \iint_{S_t(x_1, x_2, x_3)} \frac{\phi(\alpha_1, \alpha_2, \alpha_3)}{t}\, d\sigma_t \text{ (Kirchhof's formula).} \tag{5, 12}$$

Here $S_t(x_1, x_2, x_3)$ denotes the sphere with radius t and center (x_1, x_2, x_3) in the hyperplane $t=0$ where the function ϕ is prescribed, and $d\sigma_t$ denotes an element of the surface of this sphere. We assume that the function $\phi(x_1, x_2, x_3)$ is continuous and bounded together with its derivatives of order up to and including k ($k \geqslant 2$). Then, as will become apparent from formula (6, 12), the function u_ϕ will have continuous derivatives of order up to and including k.

We first show that u_ϕ, given by the formula (5, 12), satisfies the initial conditions (3, 12). That u_ϕ satisfies the first of these conditions follows from the fact that

$$\left| \iint_{S_t} \frac{\phi(\alpha_1, \alpha_2, \alpha_3)}{t}\, d\sigma_t \right| \leqslant \text{least upper bound of } |\phi| \,.\, (4\pi t^2/t),$$

which means that

$$u_\phi(t, x_1, x_2, x_3) \to 0 \quad \text{as} \quad t \to 0.$$

As for the second condition, we note that if we put

$$\alpha_k = x_k + \beta_k t,$$

the integral (5, 12) becomes

$$u_\phi(t, x_1, x_2, x_3) = \frac{t}{4\pi} \iint_{S_1} \phi(x_1 + t\beta_1, x_2 + t\beta_2, x_3 + t\beta_3)\, d\sigma_1, \tag{6, 12}$$

where the integration is carried out over the surface of the sphere S_1:

$$\beta_1^2 + \beta_2^2 + \beta_3^2 = 1, \quad d\sigma_1 = d\sigma_t/t^2$$

fixed for all values of x_1, x_2, x_3, t. Hence

$$\frac{\partial u_\phi}{\partial t}=\frac{1}{4\pi}\iint_{S_1}\phi(x_1+t\beta_1,\ x_2+t\beta_2,\ x_3+t\beta_3)\,d\sigma_1$$

$$+\frac{t}{4\pi}\iint_{S_1}\sum_{k=1}^{3}\beta_k\phi_k(x_1+t\beta_1,\ x_2+t\beta_2,\ x_3+t\beta_3)\,d\sigma_1. \qquad (7,12)$$

Here ϕ_k denotes the derivative of ϕ with respect to α_k. It is easy to see that as $t \to 0$ the first term on the right approaches $\phi(x_1, x_2, x_3)$ and the second term approaches zero; the latter is true because the integral in that term stays bounded.

It remains to show that u_ϕ, as defined by Kirchhof's formula, satisfies equation (1, 12). From equation (6, 12) we find that

$$\frac{\partial^2 u_\phi}{\partial x_1^2}+\frac{\partial^2 u_\phi}{\partial x_2^2}+\frac{\partial^2 u_\phi}{\partial x_3^2}=\frac{t}{4\pi}\iint_{S_1}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\sigma_1$$

$$=\frac{1}{4\pi t}\iint_{S_t}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\sigma_t. \qquad (8,12)$$

To compute $\partial^2 u_\phi/\partial t^2$ we rewrite the equality (7, 12) as follows:

$$\frac{\partial u_\phi}{\partial t}=\frac{u_\phi}{t}+\frac{1}{4\pi t}\iint_{S_t}\left(\frac{\partial\phi}{\partial\alpha_1}\,d\alpha_2 d\alpha_3+\frac{\partial\phi}{\partial a_2}\,d\alpha_1 d\alpha_3+\frac{\partial\phi}{\partial\alpha_3}\,d\alpha_1 d\alpha_2\right)$$

$$=\frac{u_\phi}{t}+\frac{1}{4\pi t}\iiint_{V_t}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\alpha_1 d\alpha_2 d\alpha_3$$

$$=\frac{u_\phi}{t}+\frac{I(t)}{4\pi t}, \qquad (9,12)$$

where $$I(t)=\iiint_{V_t}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\alpha_1 d\alpha_2 d\alpha_3,$$

and V_t is the surface of the sphere of radius t with center at (x_1, x_2, x_3) in the hyperplane $t=0$.

From (9, 12) we get

$$\frac{\partial^2 u_\phi}{\partial t^2}=-\frac{u_\phi}{t^2}+\frac{1}{t}\left[\frac{u_\phi}{t}+\frac{I(t)}{4\pi t}\right]-\frac{I(t)}{4\pi t^2}+\frac{1}{4\pi t}\frac{\partial I(t)}{\partial t}$$

$$=\frac{1}{4\pi t}\frac{\partial I(t)}{\partial t}. \qquad (10,12)$$

Now it is easily seen that

$$\frac{\partial I(t)}{\partial t}=\iint_{S_t}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\sigma_t. \qquad (11,12)\dagger$$

Comparing the equalities (8, 12), (10, 12) and (11, 12) we see that the function u_ϕ defined by Kirchhof's formula actually satisfies the wave equation (1, 12).

Remark. If the function $\phi_1(x_1, x_2, x_3)$ is only known to be continuous together with its first derivatives, then the function u defined by the equalities (4, 12) and (5, 12) is only a generalized solution of the Cauchy problem.

2. We now consider the special case when ϕ does not depend on x_3. It is easy to see that in that case the function u_ϕ given by Kirchhof's formula likewise does not depend on x_3 and so satisfies the equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}. \qquad (12,12)$$

In this case, we can replace the integral over the sphere S_t by the double integral taken over the intersection K_t of this sphere with the plane $\alpha_3=x_3$. Projecting a surface element $d\sigma_t$ of this sphere on the plane $\alpha_3=x_3$, we get

$$d\sigma_t=\frac{t}{[t^2-(\alpha_1-x_1)^2-(\alpha_2-x_2)^2]^{\frac{1}{2}}}\,d\alpha_1 d\alpha_2,$$

and Kirchhof's formula can then be rewritten as follows:

$$u_\phi(t,x_1,x_2)$$
$$=\frac{1}{4\pi}\iint_{S_t}\frac{\phi(\alpha_1,\alpha_2)}{t}\,d\sigma_t=\frac{1}{2\pi}\iint_{K_t}\frac{\phi(\alpha_1,\alpha_2)\,d\alpha_1 d\alpha_2}{[t^2-(\alpha_1-x_1)^2-(\alpha_2-x_2)^2]^{\frac{1}{2}}}.$$

Therefore, the solution of equation (12, 12) satisfying the conditions

$$u(0,x_1,x_2)=\phi_0(x_1,x_2),$$
$$u_t'(0,x_1,x_2)=\phi_1(x_1,x_2)$$

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† In fact, going over to polar coordinates (ρ, θ, ϕ) with origin at the center of V_t we have

$$I(t)=\int_0^t\int_0^\pi\int_0^{2\pi}\Delta\phi(r,\,\phi\;\theta)\;r^2\sin\theta\;dr\,d\theta\,d\phi,$$

$$\frac{\partial I(t)}{\partial t}=\int_0^\pi\int_0^{2\pi}\Delta\phi(t,\,\phi,\,\theta)\;t^2\sin\theta\;d\theta\,d\phi=\iint_{S_t}\Delta\phi\,d\sigma_t.$$

is given by the formula

$$u(t,x_1,x_2)=\frac{1}{2\pi}\iint_{K_t}\frac{\phi_1(\alpha_1,\alpha_2)\,d\alpha_1\,d\alpha_2}{[t^2-(\alpha_1-x_1)^2-(\alpha_2-x_2)^2]^{\frac{1}{2}}}$$
$$+\frac{1}{2\pi}\frac{\partial}{\partial t}\iint_{K_t}\frac{\phi_0(\alpha_1,\alpha_2)\,d\alpha_1\,d\alpha_2}{[t^2-(\alpha_1-x_1)^2-(\alpha_2-x_2)^2]^{\frac{1}{2}}}. \qquad (13,12)$$

This is the so-called *Poisson formula.*

3. If the function ϕ does not depend on either x_2 or x_3, then the function u_ϕ given by Kirchhof's formula is also independent of x_2 and x_3 and so satisfies the equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}. \qquad (14,12)$$

In that case it is possible to rewrite Kirchhof's formula as follows:

$$u_\phi(t,x_1)=\frac{1}{4\pi}\iint_{S_t}\frac{\phi(\alpha_1)}{t}\,d\sigma_t=\frac{1}{2}\int_{x_1-t}^{x_1+t}\phi(\alpha)\,d\alpha.$$

Here we have made use of the fact that the area of the portion of the sphere S_t contained between two planes, $\alpha_1=\text{const.}$ and $\alpha+d\alpha_1=\text{const.}$, which intersect S_t is equal to $2\pi t\,d\alpha_1$ and the function $\phi(\alpha)$ has almost constant value at all points of this portion of the sphere.[†]

Therefore the solution of equation (14, 12), satisfying the conditions

$$u(0,x_1)=\phi_0(x_1),\quad u_t'(0,x)=\phi_1(x_1),$$

is given by the formula

$$u(t,x)=\frac{1}{2}\int_{x_1-t}^{x_1+t}\phi_1(\alpha_1)\,d\alpha_1+\frac{1}{2}\frac{\partial}{\partial t}\int_{x_1-t}^{x_1+t}\phi_0(\alpha_1)\,d\alpha_1$$
$$=\frac{\phi_0(x_1+t)+\phi_0(x_1-t)}{2}+\frac{1}{2}\int_{x_1-t}^{x_1+t}\phi_1(\alpha)\,d\alpha. \qquad (15,12)$$

This formula is known as *d'Alembert's formula.*

We recall that according to the uniqueness theorem proved in § 11 there exist no solutions of the Cauchy problem other than those given by (4, 12) for equation (1, 12), by (13, 12) for equation (12, 12), and by (15, 12) for equation (14, 12). The method used in obtaining solutions of the Cauchy problem for equations (12, 12) and (14, 12) from the solution of the Cauchy problem for equation (1, 12) is called the *method of descent.*

[†] The area of a narrow slice of width $d\alpha$ of the surface of a sphere is approximately equal to $2\pi\rho\,dl$, where ρ is the radius of the middle section of the slice and dl is the length of a generator of the truncated cone inscribed in the slice. But $t/\rho=dl/d\alpha$, whence $\rho\,dl=t\,d\alpha$ and $d\sigma_t=2\pi t\,d\alpha$.

We have found the solution of the Cauchy problem for $t > 0$. The case $t < 0$ reduces to the case previously considered if we replace t by $-t$; this transformation does not change equations (1, 12), (12, 12) and (14, 12).

§ 13. Examination of the formulas which give the solution of the Cauchy problem

1. *Continuous dependence of the solution on the initial conditions.* The formulas which we derived in the preceding section and which solve the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{1, 13}$$

for $n = 2, 3$ all contain integrals of the initial functions and time derivatives of such integrals; in the case $n = 1$ they contain only integrals of the initial functions.

It follows that, if the initial functions ϕ_0 and ϕ_1 are changed so that these functions as well as their first derivatives change by a sufficiently small amount, the function $u(t, x_1, \ldots, x_n)$ which gives a solution of the Cauchy problem will also change by a small amount. In the case $n = 1$ the same conclusion holds if only the initial functions ϕ_0 and ϕ_1 are changed by a small amount. If the region in which the initial conditions are prescribed is infinite, then it is obviously necessary to assume that only values of t below a certain bound are considered.

This shows that *the Cauchy problem for equations* (1, 12), (12, 12) and (14, 12) *is reasonable.*

It is possible to derive formulas analogous to the formulas (4, 12), (13, 12) and (15, 12) for the solution of the Cauchy problem for an arbitrary n. It can then be shown that the Cauchy problem is reasonable for such an equation provided that the initial conditions are prescribed for $t = 0$.† The numbers L_1 and L_2 which appear in the definition of reasonableness (cf. § 8) are $[\frac{1}{2}n] + 2$ and $[\frac{1}{2}n]$, respectively; here $[x]$ denotes the integral part of x.

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† Such formulas can be derived by the method, employed for instance, in Goursat, *Course of Mathematical Analysis.*

Formulas (4, 12) and (13, 12) show that, regardless of how small ϕ_0 and ϕ_1 are, the values of $|u(t, x_1, x_2, x_3)|$ and $|u(t, x_1, x_2)|$, respectively, can be very large for small values of t if only the derivatives of ϕ_0 and ϕ_1 are large; the waves can 'splash'.

2. *Diffusion of waves.* Formulas (4, 12) and (5, 12) show that the value at the point $(t, x_1, \ldots, x_n)$ of the solution of the wave equation (1, 13) for $n = 3$ depends only on the initial conditions on the boundary of the base of the characteristic cone with vertex at (t, x_1, x_2, x_3). On the other hand, for $n = 1$ or $n = 2$, $u(t, x_1, \ldots, x_n)$ depends on the conditions as given on the whole base of that cone. This is clear from formulas (13, 12) and (15, 12).

Let us assume that the initial values of u and u'_t for $t = 0$ differ from zero only inside a small region G_ϵ about some point $(0, x_1^0, \ldots, x_n^0)$. Beginning with $t = 0$, we consider the values of u at the points $(t, x_1, \ldots, x_2)$ for fixed $x_1, \ldots, x_n$ and for increasing t. In the case $n = 3$, the magnitude of $u(t, x_1, \ldots, x_n)$ can differ from zero only on a small portion of the considered straight line (in the space $(t, x_1, \ldots, x_n)$); namely, on that portion of the line on which are located the vertices of those characteristic cones the boundaries of whose bases intersect the region G_ϵ. On the other hand, if $n = 1$ (or $n = 2$) and the point $(0, x_1)$ (or $(0, x_1, x_2)$) does not belong to G_ϵ, then $u(t, x_1)$ (or $u(t, x_1, x_2)$) is equal to zero for sufficiently small t and is, in general, different from zero beginning with the values of t for which the segment $|x_1 - \alpha_1| \leqslant t$ (or the circle $(\alpha_1 - x_1)^2 + (\alpha_2 - x_2)^2 \leqslant t^2$) intersects the region G_ϵ.

Consequently, a disturbance set up at $t = 0$ in some small neighborhood of the point $(x_1^0, \ldots, x_n^0)$ will, for $n = 3$ and $t > 0$, affect the values of the function only in those points of the space $(x_1, \ldots, x_n)$ which lie *close to* the sphere of radius t with center $(x_1, \ldots, x_n)$. Thus a disturbance set up at $t = 0$ at the point (x_1^0, x_2^0, x_3^0) gives rise to a spherical wave with center at that point, and this wave has a well-defined front and back. On the other hand, if $n = 1$ or $n = 2$, then a disturbance set up at $t = 0$ in a neighborhood of the point $(x_1^0, \ldots, x_n^0)$ affects, in general, *all points in the interior of the sphere with radius t and center* $(x_1^0, \ldots, x_n^0)$. The resulting wave has a well-defined front, but the back of the wave is blurred. We say that in this case the wave is diffused. For $n = 3$ no diffusion takes place. It can be shown that no diffusion takes place for the solutions of equation (1, 13) for an arbitrary odd $n \geqslant 3$.

Disturbances set up in a small region G_ϵ of a three-dimensional rigid elastic body or of a gas give rise to waves which leave no trace after them, provided the vibrations satisfy equation (1, 12). In the case of a gas, $u(t, x_1, x_2, x_3)$ denotes, for instance, the deviation from normal air pressure at the point (x_1, x_2, x_3) at the moment t. On the other hand, disturbances set up in a small region G_ϵ of a two-dimensional continuum, as, for example, a tight membrane or a water surface, give rise to waves which, in theory, leave a permanent trace after them, provided these vibrations satisfy equation (12, 12). In practice they die down very quickly because of the existence of friction which is not taken into consideration in deriving equation (12, 12). Likewise, a trace is left after the passing of a wave in a one-dimensional continuum (cf. para. 3 of this section).

3. *Examination of d'Alembert's formula.* We consider two special cases which give a clear idea of the behavior of the solution of equation (14, 12) in the general case.

We first consider the case when $\phi_1(x) \equiv 0$ and $\phi_0(x)$ is of the form represented in the top drawing in fig. 6 (the heavy continuous line). For the sake of brevity we write x instead of x_1. Then d'Alembert's formula has the form

$$u(t, x) = \tfrac{1}{2}[\phi_0(x+t) + \phi_0(x-t)].$$

To obtain the graph of $u(t, x)$ regarded, as a function of x, for some fixed positive t it is convenient to proceed in the following manner: We draw two coincident graphs (see broken line in the top drawing in fig. 6) obtained from the graph of $\phi_0(x)$ by halving the ordinates of $\phi_0(x)$. Next one of the two coincident graphs is moved as a whole a distance t to the right in the positive direction of the x-axis and the other a distance t to the left. As the last step we draw a new graph whose ordinates, for a given x, are equal to the sum of the corresponding ordinates of the two displaced graphs mentioned in the preceding sentence. Our drawing gives sample graphs $u(0, x)$, $u(\frac{1}{4}, x)$, $u(\frac{1}{2}, x)$, $u(1, x)$. (In each case the broken lines are auxiliary lines and the continuous thick lines are the graphs of $u(t, x)$ for a fixed value of t.)

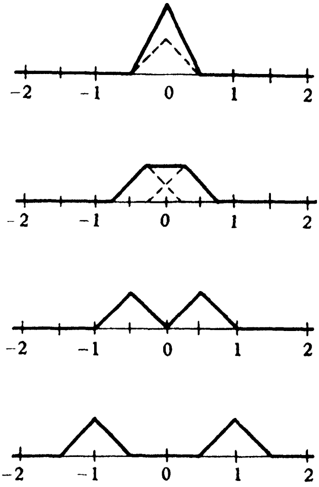

Figure 6

We now consider the case when $\phi_0(x) \equiv 0$, and

$$\phi_1(x) = \begin{cases} 1 & \text{for} \quad |x| \leqslant \frac{1}{2}, \\ 0 & \text{for} \quad |x| > \frac{1}{2}. \end{cases}$$

Then d'Almebert's formula takes the form

$$u(t,x) = \frac{1}{2}\int_{x-t}^{x+t} \phi_1(\alpha)\,d\alpha.$$

For each fixed x, $u(t,x) = 0$ as long as the interval $(-\frac{1}{2}, \frac{1}{2})$, where $\phi_1(x) \neq 0$, has no part in common with the interval $(x-t, x+t)$. The value of $u(t,x)$ changes as long as the intersection of the expanding interval $(x-t, x+t)$ with the interval $(-\frac{1}{2}, \frac{1}{2})$ continues to expand. As soon as the interval $(x-t, x+t)$ contains all of the interval $(-\frac{1}{2}, \frac{1}{2})$, the value of $u(t,x)$ stays fixed and has the value

$$\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_1(\alpha)\,d\alpha.$$

To obtain the graph representing the form of the string for different values of t, it is convenient to proceed as follows. We denote by $\Phi(z)$ a primitive function of $\phi_1(z)$. Then

$$u(t,x) = \tfrac{1}{2}[\Phi(x+t) - \Phi(x-t)].$$

To obtain the graph of $u(t,x)$ we draw the graphs of the functions $\frac{1}{2}\Phi(x)$ and $-\frac{1}{2}\Phi(x)$, and we move them in opposite directions a distance t along the x-axis. Summing the ordinates of the displaced graphs we obtain the graph of $u(t,x)$.

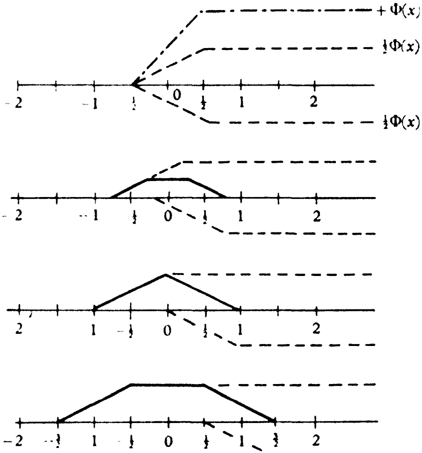

Figure 7

Fig. 7 shows the form of the string for $t = 0$, $\frac{1}{4}$, $\frac{1}{2}$ and 1.

That diffusion is taking place in this case is shown by the fact that, once out of the position of equilibrium, a point x never returns to it.

The functions $\phi_0(x)$ and $\phi_1(x)$ considered in the preceding examples are either themselves discontinuous ($\phi_1(x)$) or have discontinuous derivatives ($\phi_0(x)$). Therefore to these functions there correspond generalized solutions of equation (14, 12). To get the usual twice continuously differentiable solution of the equation, it suffices to change somewhat the functions $\phi_0(x)$ and $\phi_1(x)$ so as to get graphs of functions with continuous second derivatives. For the function $\phi_0(x)$ this can be done so that the ordinate of $\phi_0(x)$ changes by a small amount everywhere. Then the corresponding solution of equation (14, 12) changes by a small amount everywhere. $\phi_1(x)$ can be replaced by a continuous smooth function in such a manner that $\Phi(x)$ changes by an arbitrarily small amount. Then $u(t, x)$ will also change by a small amount everywhere.

§ 14. The Lorentz transformation

1. In § 1 we mentioned the fact that except for a constant multiplier the expression

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}$$

is the only linear combination of second derivatives which does not change its form under a rotation of the space, i.e. under an arbitrary orthogonal transformation of the coordinates x_1, x_2, x_3. We now consider in some detail a certain class of linear transformations of the variables (t, x_1, x_2, x_3) with constant real coefficients which are closely connected with the wave equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}-\frac{\partial^2 u}{\partial x_3^2}=0. \qquad (1, 14)$$

By a *Lorentz transformation* of the variables x_0, x_1, x_2, x_3 we mean any linear homogeneous transformation of these variables

$$y_i=\sum_{j=0}^{3} a_{ij}x_j \quad (i=0, 1, 2, 3) \qquad (2, 14)$$

with real coefficients a_{ij} which leaves invariant the quadratic form

$$x_0^2-x_1^2-x_2^2-x_3^2. \qquad (3, 14)$$

This means that in the new variables this quadratic form becomes

$$y_0^2-y_1^2-y_2^2-y_3^2.$$

It is easy to check that the totality of Lorentz transformations forms a group. In particular, it is easy to see that the product of two Lorentz transformations (substitutions) is again a Lorentz transformation.

We now write down a formula for a special class of Lorentz transformations which have the property of leaving invariant two of the last three (space) coordinates. Such transformation has the form

$$\left.\begin{aligned} y_0 &= \alpha x_0 + \beta x_1, \\ y_1 &= \gamma x_0 + \delta x_1, \\ y_2 &= x_2, \\ y_3 &= x_3. \end{aligned}\right\} \tag{4, 14}$$

For such transformations the identity

$$y_0^2 - y_1^2 \equiv x_0^2 - x_1^2$$

must hold. Substituting in this identity the expressions for y_0 and y_1 in (4, 14), we get

$$(\alpha x_0 + \beta x_1)^2 - (\gamma x_0 + \delta x_1)^2 \equiv x_0^2 - x_1^2.$$

Whence

$$\left.\begin{aligned} \alpha^2 - \gamma^2 &= 1, \\ \beta^2 - \delta^2 &= -1, \\ \alpha\beta - \gamma\delta &= 0. \end{aligned}\right\} \tag{5, 14}$$

In particular, these equations are satisfied if we put

$$\alpha = \delta = \cosh \psi,$$
$$\beta = \gamma = \sinh \psi,$$

where ψ is an arbitrary number. Then

$$\begin{aligned} y_0 &= \cosh \psi x_0 + \sinh \psi x_1, \\ y_1 &= \sinh \psi x_0 + \cosh \psi x_1, \\ y_2 &= x_2, \\ y_3 &= x_3. \end{aligned}$$

Let us put $\tanh \psi = \beta$. We then obtain the usual formulas for the class of Lorentz transformations under consideration:

$$\left.\begin{aligned} y_0 &= (x_0 + \beta x_1)/(1-\beta^2)^{\frac{1}{2}}, \\ y_1 &= (\beta x_0 + x_1)/(1-\beta^2)^{\frac{1}{2}}, \\ y_2 &= x_2, \\ y_3 &= x_3. \end{aligned}\right\} \tag{6, 14}$$

Here β is an arbitrary number smaller than one in absolute value because $|\tanh \psi| < 1$ for any ψ.

The formulas (6, 14) are of fundamental importance, since, as we are going to show, *any Lorentz transformation is a combination of an orthogonal transformation of the variables* x_1, x_2, x_3 *which leaves* x_0 *fixed, a transformation of the form* (6, 14), *and a possible change of sign of one of the variables* (*a reflexion*).

Let a Lorentz transformation be given by the formulas

$$\left.\begin{aligned} y_0&=a_{00}x_0+a_{01}x_1+a_{02}x_2+a_{03}x_3,\\ y_1&=a_{10}x_0+a_{11}x_1+a_{12}x_2+a_{13}x_3,\\ y_2&=a_{20}x_0+a_{21}x_1+a_{22}x_2+a_{23}x_3,\\ y_3&=a_{30}x_0+a_{31}x_1+a_{32}x_2+a_{33}x_3.\end{aligned}\right\} \tag{7, 14}$$

We now take the variables x_1, x_2, x_3 into the variables x_1', x_2', x_3' by means of an orthogonal transformation such that

$$a_{01}x_1+a_{02}x_2+a_{03}x_3=ax_1',$$

with suitable definitions for x_2' and x_3'. If, in addition, we put x_0' equal to x_0, then, as can easily be seen, the transformation which takes x_0, x_1, x_2, x_3 into x_0', x_1', x_2', x_3' represents a Lorentz transformation. Putting the variables x_0', x_1', x_2', x_3' on the right side of (7, 14), we obtain

$$\left.\begin{aligned} y_0&=a_{00}x_0'+ax_1',\\ y_1&=a_{10}x_0'+b_{11}x_1'+b_{12}x_2'+b_{13}x_3',\\ y_2&=a_{20}x_0'+b_{21}x_1'+b_{22}x_2'+b_{23}x_3',\\ y_3&=a_{30}x_0'+b_{31}x_1'+b_{32}x_2'+b_{33}x_3'.\end{aligned}\right\} \tag{8, 14}$$

We will show that in this case $a^2<a_{00}^2$. In fact, since (8, 14) is a Lorentz transformation,

$$y_0^2-y_1^2-y_2^2-y_3^2=x_0'^2-x_1'^2-x_2'^2-x_3'^2,$$

whence

$$y_1^2+y_2^2+y_3^2=x_1'^2+x_2'^2+x_3'^2-x_0'^2+y_0^2. \tag{9, 14}$$

Let us put $y_0=0$. Then $x_0'=-(a/a_{00})x_1'$, and the identity (9, 14) becomes an identity in three variables

$$y_1^2+y_2^2+y_3^2=\left(1-\frac{a^2}{a_{00}^2}\right)x_1'^2+x_2'^2+x_3'^2.$$

Since the right side is positive for any x_1', x_2', x_3',

$$1-(a^2/a_{00}^2)>0,$$

that is, $a^2<a_{00}^2$.

Let us put $a/a_{00}=\beta$ and let us carry out a Lorentz transformation of the form (6, 14):

$$\left.\begin{aligned} x_0''&=(x_0'+\beta x_1')/(1-\beta^2)^{\frac{1}{2}},\\ x_1''&=(\beta x_0'+x_1')/(1-\beta^2)^{\frac{1}{2}},\\ x_2''&=x_2',\\ x_3''&=x_3'.\end{aligned}\right\} \tag{10, 14}$$

Clearly, the connection between y_0, y_1, y_2, y_3 and $x_0'', x_1'', x_2'', x_3''$ is given by a Lorentz transformation of the form

$$\left.\begin{aligned} y_0 &= cx_0'', \\ y_1 &= c_{10}x_0'' + c_{11}x_1'' + c_{12}x_2'' + c_{13}x_3'', \\ y_2 &= c_{20}x_0'' + c_{21}x_1'' + c_{22}x_2'' + c_{23}x_3'', \\ y_3 &= c_{30}x_0'' + c_{31}x_1'' + c_{32}x_2'' + c_{33}x_3''. \end{aligned}\right\} \tag{11, 14}$$

where, as can easily be computed, $c = (a_{00}^2 - a^2)^{\frac{1}{2}}$.

We will find the values of the coefficients c, c_{10}, c_{20} and c_{30}.

Putting $x_0'' = 1, x_1'' = x_2'' = x_3'' = 0$, we obtain

$$y_0 = c, \quad y_1 = c_{10}, \quad y_2 = c_{20}, \quad y_3 = c_{30}.$$

Hence $$1 = c^2 - c_{10}^2 - c_{20}^2 - c_{30}^2 \quad \text{and} \quad c^2 \geqslant 1.$$

Putting $y_0 = 1$, $y_1 = y_2 = y_3 = 0$, we find that $x_0'' = 1/c$ and that x_1'', x_2'', x_3'' have some definite values $\tilde{x}_1'', \tilde{x}_2'', \tilde{x}_3''$. It follows that

$$1 = c^{-2} - \tilde{x}_1''^2 - \tilde{x}_2''^2 - \tilde{x}_3''^2 \quad \text{and} \quad c^{-2} \geqslant 1,$$

that is, $c^2 \leqslant 1$.

Consequently, $c^2 = 1$, and, going back to the equality

$$1 = c^2 - c_{10}^2 - c_{20}^2 - c_{30}^2,$$

we see that $$c_{10} = c_{20} = c_{30} = 0.$$

Consequently, the transformation (11, 14) has, in fact, the form

$$\left.\begin{aligned} y_0 &= \pm x_0'', \\ y_1 &= \qquad c_{11}x_1'' + c_{12}x_2'' + c_{13}x_3'', \\ y_2 &= \qquad c_{21}x_1'' + c_{22}x_2'' + c_{23}x_3'', \\ y_3 &= \qquad c_{31}x_1'' + c_{32}x_2'' + c_{33}x_3''. \end{aligned}\right\} \tag{12, 14}$$

Changing, if necessary, the sign of the coordinate x_0'', we obtain a Lorentz transformation which is simply an orthogonal transformation of the variables x_1'', x_2'', x_3'' into y_1, y_2, y_3.

Thus we see that the most general Lorentz transformation (7, 14), which takes the variables x_i into the variables y_i, is the result of the following consecutive transformations: an orthogonal transformation which takes the x_i into the x_i', the particular Lorentz transformation (6, 14) which takes the x_i' into the x_i'', a possible change of sign of x_0'', and, finally, an orthogonal transformation of the x_i'' into the y_i $(i = 1, 2, 3)$.

If we transpose the matrix of each of the intermediate transformations, we will again obtain a matrix of a transformation of the same type. It follows from this that *the transpose of a matrix of a Lorentz transformation is again a matrix of a Lorentz transformation.* From the definition of a Lorentz transformation it follows that the inverse of a Lorentz transformation is also a Lorentz transformation.

2. We now prove a fundamental fact which clarifies the close connection between Lorentz transformations and the wave equation.

THEOREM. *Every non-singular linear transformation of the variables* t, x_1, x_2, x_3 *with real constant coefficients which does not change the form of equation* (1, 14) *is a combination of a Lorentz transformation, a translation of the origin in the space* (t, x_1, x_2, x_3), *and a similarity transformation in that space.*

To simplify notation we put $t = x_0$.

By the statement that some transformation 'does not change the form of the equation' we mean that any function $u(x_0, x_1, x_2, x_3)$ (with continuous second derivatives) which satisfies the equation

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = 0$$

goes over, after transformation of the x_i into the y_i, into a function $u(y_0, y_1, y_2, y_3)$ satisfying the equation

$$\frac{\partial^2 u}{\partial y_0^2} - \frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} - \frac{\partial^2 u}{\partial y_3^2} = 0. \qquad (13, 14)$$

It follows that for any such transformation the following equality holds for an arbitrary function $u(x_0, x_1, x_2, x_3)$:

$$\frac{\partial^2 u}{\partial y_0^2} - \frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} - \frac{\partial^2 u}{\partial y_3^2} = k\left(\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}\right), \qquad (14, 14)$$

where k is an arbitrary constant. In fact, if we assume, most generally, that

$$\frac{\partial^2 u}{\partial y_0^2} - \frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} - \frac{\partial^2 u}{\partial y_3^2} \equiv \sum_{i,j=0}^{3} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad (15, 14)$$

and that
$$\sum_{i,j=0}^{3} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \neq k\left(\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}\right),$$

we will arrive at a contradiction. What will be contradicted is the fact that, as a result of such a transformation of the variables, every solution of equation (1, 14) goes over into a solution of a like equation. Indeed, under our assumption, it is possible to choose a system of numbers u^0_{ik} satisfying the two linear equations

$$\sum_{i,j=0}^{3} A_{ij} u^0_{ij} = 1, \tag{16, 14)$_1$}$$

$$u^0_{00} - u^0_{11} - u^0_{22} - u^0_{33} = 0. \tag{16, 14)$_2$}$$

The function $$u(x_0, x_1, x_2, x_3) = \frac{1}{2} \sum_{i,j=0}^{3} u^0_{ij} x_i x_j$$

satisfies the equations $\partial^2 u / \partial x_i \partial x_j = u^0_{ij}$.

In view of (16, 14) this function satisfies equation (1, 14) and, after transformation of the variables, it will not satisfy equation (13, 14). The latter conclusion follows from (16, 14) and (15, 14). Consequently, (14, 14) holds.

Let us carry out the similarity transformation

$$x'_i = x_i \,|\, k \,|^{-\frac{1}{2}}.$$

Then $$k\left(\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}\right) = \pm\left(\frac{\partial^2 u}{\partial x_0'^2} - \frac{\partial^2 u}{\partial x_1'^2} - \frac{\partial^2 u}{\partial x_2'^2} - \frac{\partial^2 u}{\partial x_3'^2}\right).$$

Consequently, it is sufficient to prove that a transformation

$$y_i = \sum_{j=0}^{3} a_{ij} x'_j \quad (i = 0, \ldots, 3), \tag{17, 14}$$

which does not change the absolute value of the differential expression

$$\frac{\partial^2 u}{\partial x_0'^2} - \frac{\partial^2 u}{\partial x_1'^2} - \frac{\partial^2 u}{\partial x_2'^2} - \frac{\partial^2 u}{\partial x_3'^2}, \tag{18, 14}$$

is a Lorentz transformation, that is, it does not change the quadratic form

$$x_0'^2 - x_1'^2 - x_2'^2 - x_3'^2. \tag{19, 14}$$

But this follows from the fact that, as was shown in § 5, under a linear transformation of the independent variables of the form (17, 14) the expression,

$$\sum_{i,j=0}^{3} c_{ij} (\partial^2 u / \partial x'_i \partial x'_j)$$

transforms like the quadratic form

$$\sum_{i,j=0}^{3} c_{ij} x'_i x'_j$$

when the latter is subjected to the transformation

$$x'_j = \sum_{i=0}^{3} a_{ij} y_i \quad (j = 0, \ldots, 3). \tag{20, 14}$$

Since the transformation (17, 14) does not change the form of the expression (18, 14), with the possible exception of sign, the transformation (20, 14) does not change the form of the quadratic form (19, 14), with the possible exception of sign. According to the law of inertia, the sign of (19, 14) cannot, however, change under any transformation with real coefficients. Consequently, the transformation (20, 14) is a Lorentz transformation. Thus, according to para. 1, the initial transformation (17, 14) is a Lorentz transformation, for its matrix is the transposed inverse of the matrix of the Lorentz transformation (20, 14).

We have thus shown that each homogeneous linear transformation which does not change the form of equation (1, 14) is a combination of a similarity transformation and a Lorentz transformation. Since it is obvious that the translation of the origin does not change the form of this equation, our theorem has been completely proved.

3. Consider an arbitrary hyperplane through the origin in the space t, x_1, x_2, x_3 forming an angle greater than 45° with the t-axis. Such hyperplane (and only such) can be taken into a hyperplane[†]

$$t = \beta x_1, \quad \text{where} \quad |\beta| < 1$$

by means of an orthogonal transformation of the variables x_1, x_2, x_3. Using the Lorentz transformation (6, 14), we can take this hyperplane into the coordinate hyperplane $t^* = 0$. Consequently, it is always possible to take an arbitrary hyperplane [in the space (t, x_1, x_2, x_3)] forming an angle greater than 45° with the t-axis into the hyperplane $t = 0$ by means of a linear transformation of the

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[†] Let the equation of the original hyperplane be of the form

$$At + Bx_1 + Cx_2 + Dx_3 = 0, \quad B^2 + C^2 + D^2 = 1.$$

Then the cosine of the angle α_0 between the normal to the hyperplane and the t-axis is $A/(A^2+1)^{\frac{1}{2}}$, and its tangent is A^{-1}. Since the normal to the hyperplane forms an acute angle with the t-axis, the mapping

$$Bx_1 + Cx_2 + Dx_3 = x'_1,$$

with suitably (i.e. in accordance with the orthogonality property of the transformation) chosen x'_2 and x'_3, maps the given hyperplane into a hyperplane of the form

$$At + x'_1 = 0 \quad \text{or} \quad t = -(A^{-1})x'_1, \quad |A^{-1}| < 1.$$

independent variables which does not change the form of equation (1, 14). This enables us to solve the Cauchy problem for equation (1, 14) with initial conditions prescribed on an arbitrary hyperplane π forming an angle $> 45°$ with the t-axis instead of on the hyperplane $t = 0$. This is equivalent to saying that the initial conditions may be prescribed on a hyperplane π which cuts either the lateral surface or the vertex of each characteristic cone of equation (1, 14). In fact, by prescribing the values of the function u and its derivative in an arbitrary direction not in π at all points of some region G_0 in π, we automatically prescribe in G_0 the values of the first derivative of u in every other direction in the space (t, x_1, x_2, x_3). This follows from the fact that, if we know the values of u in G_0, we also know the values of its first derivative in any direction which lies in G_0. And, by transforming the hyperplane π into the hyperplane $t^* = 0$, we reduce the Cauchy problem with initial data on π to the Cauchy problem considered in § 12.

On the other hand, it is easy to show that the Cauchy problem for equation (1, 14) is not reasonable if the initial conditions are prescribed on a hyperplane π forming, in the space $t, x_1 \ldots, x_n$, an angle $\leqslant 45°$ with the t-axis.

In fact, if the hyperplane π forms a 45° angle with the t-axis, it is a characteristic surface, and the Cauchy conditions cannot be arbitrarily prescribed on it regardless of their degree of smoothness.

We now consider the case when π forms an angle $< 45°$ with the t-axis. By means of an orthogonal transformation of the coordinates in the space (x_1, x_2, x_3) and by means of a parallel translation it is possible to transform π into a plane of the form

$$\beta t' + x_1' = 0, \quad |\beta| < 1.$$

As already observed, such transformations have no effect on the form of equation (1, 14). Further, by means of a Lorentz transformation, it is possible to give the plane π the form

$$x_1^* = 0.$$

The form of equation (1, 14) remains unchanged by this transformation.

We now prescribe the following Cauchy conditions on the hyperplane $x_1^* = 0$:

$$\left.\begin{aligned} u(t^*, 0, x_2^*, x_3^*) &= \phi_0(x_2^*), \\ u'_{x_1^*}(t^*, 0, x_2^*, x_3^*) &= \phi_1(x_2^*). \end{aligned}\right\} \qquad (21, 14)$$

If it exists at all, the solution $u(t^*, x_1^*, x_2^*, x_3^*)$ of this Cauchy problem is independent of t^* and of x_3^*. In fact, let

$$(\alpha, x_1^*, x_2^*, a) \quad \text{and} \quad (\beta, x_1^*, x_2^*, b)$$

be two points such that the pair of numbers (α, a) does not coincide with the pair of numbers (β, b), and let u be assumed to take on different values at these two points. Then the function

$$u(t^*, x_1^*, x_2^*, x_3^*) - u(t^* + \gamma, x_1^*, x_2^*, x_3^* + c),$$

where
$$\gamma = \beta - \alpha \quad \text{and} \quad c = b - a,$$

satisfies equation (1, 14) and, without being identically zero, vanishes for $x_1^* = 0$ together with its first derivative with respect to x_1^*. This contradicts Holmgren's theorem concerning the uniqueness of the solution of Cauchy's problem for equation (1, 14) with constant and, consequently, analytic coefficients.

We have thus shown that, if there exists a solution of the Cauchy problem for equation (1, 14) with initial data of the form (21, 14), then this solution is independent of t^* and of x_3^*. But then this solution satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x_1^{*2}} + \frac{\partial^2 u}{\partial x_2^{*2}} = 0,$$

for which we have already proved the unreasonableness of the Cauchy problem.

§ 15. The mathematical foundations of the special principle of relativity

The special principle of relativity asserts that all laws of nature have the same form for all observers who move with respect to one another with uniform, straight-line motion. More precisely, for each of these observers there exists a 'local' space-time coordinate system (t, x_1, x_2, x_3) (t is the time coordinate, and x_1, x_2, x_3 are the space coordinates) in which any given natural law is expressible by means of the same equations. In particular, for each of these observers the speed of light in his local coordinate system is the same in all directions. For the sake of simplicity we assume that this speed is equal to 1.

We wish to find a connexion between the local space-time coordinates of two observers A' and A'' of whom A'' is assumed to be moving uniformly and along a straight line with respect to A'. The velocity of A'' is equal to β, $|\beta|<1$. In view of the assumption that space and time are homogeneous and isotropic, we consider the required connexion to be linear and its coefficients to be functions of β only. We denote the local space-time coordinates of A' by (t', x_1', x_2', x_3') and the local space-time coordinates of A'' by $(t'', x_1'', x_2'', x_3'')$. For simplicity in notation we shall sometimes write x_0' instead of t' and x_0'' instead of t''.

Thus, let

$$x_i''=\sum_{j=0}^{3} a_{ij}(\beta)\, x_i' + \alpha_i \quad (i=0,1,2,3). \tag{1, 15}$$

We shall find the connexion between the coordinates (t, x_1', x_2', x_3') and the coordinates $(t'', x_1'', x_2'', x_3'')$, using only the assumption that the velocity of light is constant for the observers A' and A''.

We describe the rectilinear propagation of a plane light wave by means of some non-constant function

$$f(a_0 t' + a_1 x_1' + a_2 x_2' + a_3 x_3'), \tag{2, 15}$$

whose values move as a result of the changing time t' in a direction perpendicular to the plane

$$a_1 x_1' + a_2 x_2' + a_3 x_3' = \text{const.}$$

with the speed

$$-a_0/(a_1^2+a_2^2+a_3^2)^{\frac{1}{2}}$$

which is equal to 1 by assumption. Here a_0, a_1, a_2, a_3 are constants. It follows that

$$a_0^2 = a_1^2 + a_2^2 + a_3^2. \tag{3, 15}$$

Since the velocity of light for the observer A'' in his local coordinates t'', x_1'', x_2'', x_3'' is also equal to 1, we find, on going from the coordinates x_i' to the coordinates x_i'', that the expression

$$a_0 t' + a_1 x_1' + a_2 x_2' + a_3 x_3'$$

goes over into the expression

$$a_0' t'' + a_1' x_1'' + a_2' x_2'' + a_3' x_3'' + b$$

and

$$a_0'^2 = a_1'^2 + a_2'^2 + a_3'^2. \tag{4, 15}$$

We shall show that the coordinates t'', x_1'', x_2'', x_3'' are obtained from the coordinates t', x_1', x_2', x_3' by means of a Lorentz transformation and a translation of the origin. By means of a translation of the origin we can replace the coordinates t'', x_1'', x_2'', x_3'' by coordinates

t, x_1, x_2, x_3 whose connexion with t', x_1', x_2', x_3' is given by the homogeneous linear equations

$$x_i = \sum_{j=0}^{3} a_{ij}(\beta) x_j' \quad (i = 0, 1, 2, 3). \tag{5, 15}$$

Now let the function

$$f(a_0 x_0' + a_1 x_1' + a_2 x_2' + a_3 x_3')$$

go over into the function

$$f(a_0' x_0 + a_1' x_1 + a_2' x_2 + a_3' x_3),$$

where the numbers a_0', a_1', a_2', a_3' satisfy the relation

$$a_0'^2 - a_1'^2 - a_2'^2 - a_3'^2 = 0$$

as soon as the numbers a_0, a_1, a_2, a_3 satisfy the relation

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = 0.$$

Here a_0, a_1, a_2, a_3 is an *arbitrary* system of numbers satisfying equation (3, 15), and a_0', a_1', a_2', a_3' is the corresponding system of numbers resulting from the transformation (5, 15). We shall show that it now follows that (5, 15) yields the Lorentz transformation for the coefficients a, that is, that

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = a_0'^2 - a_1'^2 - a_2'^2 - a_3'^2.$$

In fact, what we do know from the form of the substitution (5, 15) is that the relation between the a_i and their transforms under (5, 15) must be of the form

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 \equiv \sum_{i,j=0}^{3} k_{ij}(\beta) a_i' a_j'. \tag{6, 15}$$

We first show that

$$\sum_{i,j=0}^{3} k_{ij}(\beta) a_i' a_j' \equiv k(\beta)(a_0'^2 - a_1'^2 - a_2'^2 - a_3'^2). \tag{7, 15}$$

In fact,

$$\Sigma k_{ij}(\beta) a_i' a_j' = 0 \tag{8, 15}$$

implies

$$a_0'^2 - a_1'^2 - a_2'^2 - a_3'^2 = 0, \tag{9, 15}$$

and conversely. This means that the surfaces given in the four-dimensional space a_0', a_1', a_2', a_3' by (8, 15) and (9, 15) coincide. It is easy to see that this implies the correctness of (7, 15). Consequently

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = k(\beta)(a_0'^2 - a_1'^2 - a_2'^2 - a_3'^2).$$

If we consider the motion of the first system with respect to the second, we have, analogously,

$$a_0'^2 - a_1'^2 - a_2'^2 - a_3'^2 = k(-\beta)(a_0^2 - a_1^2 - a_2^2 - a_3^2),$$

whence
$$k(\beta)\, k(-\beta) = 1.$$

On the other hand, since neither system is in any way distinguished, we must have
$$k(\beta) = k(-\beta),$$

and, consequently, $k(\beta) = \pm 1$.

When the variables x_i' are subjected to the transformation (5, 15), the variables a_i are also subjected to a linear transformation. It follows that the number of plus and minus signs in the quadratic form in the a_i cannot change. Hence $k(\beta) = 1$, and the form

$$a_0^2 - a_1^2 - a_2^2 - a_3^2$$

must remain invariant under the transformation (5, 15). Thus, the transformation of the variables a_i is a Lorentz transformation. The linear transformation to which the a_i are subjected when the x_i are transformed by means of (5, 15) is given by a matrix which is the transposed inverse of the matrix (5, 15). Consequently, (5, 15) is itself a Lorentz transformation (cf. para. 1, § 14), which is what we set out to prove.

§ 16. Survey of the fundamental facts of the theory of the Cauchy problem for general hyperbolic systems

So far we have discussed the Cauchy problem for the wave equation (1, 12). In this section we give a brief survey of the basic facts of the theory of the Cauchy problem for general hyperbolic equations without giving proofs of these facts. Of primary interest to us are linear second-order equations.

1. Consider the linear equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} A_{0i} \frac{\partial^2 u}{\partial t\, \partial x_i} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + B_0 \frac{\partial u}{\partial t} + Cu + D, \tag{1, 16}$$

whose coefficients A_{ij}, A_{0i}, B_i, C and D are functions of $t, x_1, \dots, x_n$. We call this equation t-hyperbolic in some region G of the space $(t, x_1, \dots, x_n)$ if the following condition holds. Every straight line in the real space $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ going through the origin cuts the surface

$$1 = \sum_{i,j=1}^{n} A_{ij}(t, x_1, \ldots, x_n)\,\alpha_i \alpha_j + \sum_{i=1}^{n} A_{0i}(t, x_1, \ldots, x_n)\,\alpha_i \qquad (2, 16)$$

in two real distinct points. If $\alpha_1, \ldots, \alpha_n$ satisfy equation (2, 16), then the hyperplane (in the space $(t, x_1, \ldots, x_n)$) whose normal is parallel to the vector $(1, \alpha_1, \ldots, \alpha_n)$ is characteristic (cf. § 3).

Let K be a surface with a conical singular point at $t = t^0$, $x_i = x_i^0$. We call K a characteristic cone of equation (1, 16) at the point $(t^0, x_1^0, \ldots, x_n^0)$ if the tangent hyperplane at every point of K (including the vertex) is characteristic.

If

$$F(t, x_1, \ldots, x_n) = 0$$

is the equation of the surface of the characteristic cone (or, more generally, of any characteristic surface (cf. § 3)), the function F must satisfy the equation

$$\left(\frac{\partial F}{\partial t}\right)^2 = \sum_{i,j=1}^{n} A_{ij} \frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j} + \sum_{i=1}^{n} A_{0i} \frac{\partial F}{\partial t}\frac{\partial F}{\partial x_i}.$$

At each point $(t^0, x_1^0, \ldots, x_n^0)$ in the region G at which equation (1, 16) is t-hyperbolic there exists in G a unique characteristic cone with vertex at this point which intersects every hyperplane $t =$ const. in some closed surface S, provided that $|t - t^0|$ is sufficiently small. This cone and the part of the hyperplane $t =$ const. which is bounded by the surface S bound a certain region K.

If $n = 1$, the characteristic cone degenerates into two curves l_1 and l_2 emanating from the point (t^0, x^0), and its base degenerates into the straight-line segment whose end-points are the points of intersection of the line $t =$ const. with the curves l_1 and l_2.

2. There exists a number L depending on n with the property that, for every pair of L times continuously differentiable functions $\phi_0(x_1, \ldots, x_n)$ and $\phi_1(x_1, \ldots, x_n)$ prescribed in some region G_0 of the hyperplane $t = t_0$, the t-hyperbolic equation (1, 16) has a unique twice continuously differentiable solution satisfying the conditions

$$\left.\begin{aligned} u(t_0, x_1, \ldots, x_n) &= \phi_0(x_1, \ldots, x_n), \\ u_t'(t_0, x_1, \ldots, x_n) &= \phi_1(x_1, \ldots, x_n). \end{aligned}\right\} \qquad (3, 16)$$

Given a point $(t, x_1, \ldots, x_n)$, this solution is uniquely determined by the conditions (3, 16) at that point if the base of the characteristic cone with vertex at this point is entirely in G_0. We denote the totality of such points $(t, x_1, \dots, x_n)$ by G.

If the functions $\phi_0(x_1, \dots, x_n)$ and $\phi_1(x_1, \dots, x_n)$ and all their derivatives up to order L change by a sufficiently small amount, then the corresponding solution will change little in the whole region G. Hence, the Cauchy problem for equation (1, 16) is reasonable.

For linear hyperbolic equations with constant coefficients containing only terms with second derivatives, $L = [\frac{1}{2}n] + 2$. S. L. Sobolev has shown that for general second-order equations $L = [\frac{1}{2}n] + 3$; here it is assumed that the coefficients of the equation have continuous derivatives of order up to $[\frac{1}{2}n] + 2$.†

3. We say that no diffusion of waves takes place for equation (1, 16) in the considered region G in the space $(t, x_1, \dots, x_n)$ if the value at the vertex $(t, x_1, \dots, x_n)$ of the characteristic cone of the solution u of the Cauchy problem depends only on the values of ϕ_0 and ϕ_1 on the periphery of the base of this cone for an arbitrary disposition of the characteristic cone in the interior of the region G. Otherwise we say that diffusion of waves does take place. Hadamard‡ showed a long time ago that diffusion of waves always takes place for $n = 1$ and for an even n. In 1939 Matisson§ investigated the case $n = 3$. He found that, apart from insignificant transformations, for $n = 3$ the hyperbolic equations for which no diffusion of waves takes place are the same as equation (1, 12); all these equations can be obtained from equation (1, 12) by means of the following simple transformations:

(*a*) change of independent variables,

(*b*) linear substitution of the function u (i.e. replacement of u by $\alpha u + \beta$),

(*c*) multiplication of both sides of the equation by some function of $t, x_1, \dots, x_n$.

4. In this section we have considered so far only the case when the Cauchy conditions are prescribed on the hyperplane $t = 0$. The case when the Cauchy conditions are prescribed on an arbitrary surface can be reduced to this special case by a change of the

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† S. L. Sobolev, 'A new method for the solution of the Cauchy problem for linear hyperbolic equations', *Matematichesky sbornik*, vol. I (43), I (1936), pp. 39–72 (in Russian).

‡ Hadamard, *Le problème de Cauchy*, Paris (1932), pp. 209–41.

§ Matisson, *Acta Math.* vol. LXXI, no. 3–4, 1939, p. 249.

ndependent variables, provided that all characteristic cones with ertices sufficiently close to this surface intersect it in closed $n-2)$-dimensional surfaces.

5. The non-linear equation

$$\frac{\partial^2 u}{\partial t^2} = F\left(t, x_1, \dots, x_n, u, \frac{\partial u}{\partial t}, \dots, \frac{\partial u}{\partial x_i}, \dots, \frac{\partial^2 u}{\partial x_i \partial x_j}, \dots, \frac{\partial^2 u}{\partial t \partial x_j}, \dots\right) \tag{4, 16}$$

s said to be t-hyperbolic in the neighborhood of some function $(t, x_1, \dots, x_n)$ in some region G of the space $(t, x_1, \dots, x_n)$ if the ollowing linear equation is t-hyperbolic in G:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\substack{i,j=1 \\ j \geqslant i}}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} A_{0j} \frac{\partial^2 u}{\partial t \partial x_j}, \tag{5, 16}$$

here A_{ij} are the partial derivatives of the right side of equation 4, 16) with respect to $\partial^2 u/\partial x_i \partial x_j$ computed for

$$u \equiv u_0(t, x_1, \dots, x_n) \quad (i = 0, 1, \dots, n;\ j = 1, 2, \dots, n).$$

For the non-linear equation (4, 16), the Cauchy problem is easonable if the conditions

$$u(t_0, x_1, \dots, x_n) = \phi_0(x_1, \dots, x_n),$$
$$u_t'(t_0, x_1, \dots, x_n) = \phi_1(x_1, \dots, x_n),$$

rescribed for $t = t_0$, are such that equation (5, 16) is hyperbolic in he neighborhood of the function

$$u_0(t, x_1, \dots, x_n) = \phi_0(x_1, \dots, x_n) + (t - t_0)\,\phi_1(x_1, \dots, x_n).$$

S. L. Sobolev showed† that for a non-linear hyperbolic equation $L \leqslant [\frac{1}{2}n] + 4$; here it is assumed that the function F on the right side f (4, 16) has continuous derivatives of order up to $[\frac{1}{2}n] + 3$ with espect to all its arguments.

6. The system of linear equations

$$\sum_{=1}^{N} \sum_{k_0 + k_1 + \dots + k_n \leqslant n_j} A_{ij}^{(k_0 k_1 \dots k_n)}(t, x_1, \dots, x_n) \frac{\partial^k u_j}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_n^{k_n}} + \dots$$
$$= f_i(t, x_1, \dots, x_n) \quad (i = 1, 2, \dots, N)$$

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† S. L. Sobolev, 'On the Cauchy problem for quasi-linear hyperbolic equations', *C.R. Acad. Sci. U.R.S.S.*, vol. xx, no. 2–3 (1938), pp. 79–83 (in Russian). S. L. Sobolev, 'A contribution to the theory of non-linear hyperbolic partial differential equations', *Matematichesky sbornik*, vol. v (47): 1 (1939), pp. 71–99 (in Russian).

is said to be t-hyperbolic at the point $(t^0, x_1^0, \dots, x_n^0)$ if for arbitrary real α_i, the sum of whose squares is positive, the roots λ of the determinant

$$\left| \sum_{k_0+k_1+\dots+k_n=n_j} A_{ij}^{(k_0 k_1 \dots k_n)}(t^0, x_1^0, \dots, x_n^0)\, \lambda^{k_0} \alpha_1^{k_1} \dots \alpha_n^{k_n} \right|$$

are all real and distinct. t-hyperbolicity of a non-linear system in the neighborhood of any of its solutions is defined in an analogous manner.

The Cauchy problem has been proved reasonable for hyperbolic systems.†

7. We now outline a method for finding an approximate solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{6, 16}$$

with initial conditions

$$u(0, x, y) = \phi(x, y), \quad u_t'(0, x, y) = \psi(x, y).$$

The functions $\phi(x, y)$ and $\psi(x, y)$ are assumed to be four times continuously differentiable and to be defined in a square G

$$a < x < b, \quad c < y < d.$$

We construct three families of parallel planes in the space (t, x, y):

$$t = k\Delta \quad (k = 0, 1, 2, 3, \dots),$$
$$x = m\delta, \quad y = n\delta.$$

Here Δ and δ are certain positive numbers. The numbers m and n assume all integral values satisfying the inequalities

$$a < m\delta < b \quad \text{and} \quad c < n\delta < d.$$

We assume for the sake of simplicity that

$$a = m_1\delta, \quad b = m_2\delta, \quad c = n_1\delta, \quad d = n_2\delta.$$

In equation (6, 16) we replace $u_{tt}''(k\Delta, m\delta, n\delta)$ by

$$\frac{\bar{u}[(k-1)\Delta, m\delta, n\delta] + \bar{u}[(k+1)\Delta, m\delta, n\delta] - 2\bar{u}(k\Delta, m\delta, n\delta)}{\Delta^2},$$

$u_{xx}''(k\Delta, m\delta, n\delta)$ by

$$\frac{\bar{u}[k\Delta, (m+1)\delta, n\delta] + \bar{u}[k\Delta, (m-1)\delta, n\delta] - 2\bar{u}(k\Delta, m\delta, n\delta)}{\delta^2},$$

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† I. G. Petrovsky, *Matematichesky sbornik*, vol. II (44) (1937), pp. 815–70 (in Russian).

and u''_{yy} $(k\Delta, m\delta, n\delta)$ by

$$\frac{\bar{u}[k\Delta, m\delta, (n+1)\delta] + \bar{u}[k\Delta, m\delta, (n-1)\delta] - 2\bar{u}(k\Delta, m\delta, n\delta)}{\delta^2}.$$

It is easy to check that if $u(t, x, y)$ is twice continuously differentiable, then, for small values of Δ and δ, the errors which result from the above replacements are small. The differential equation (6, 16) now becomes a difference equation which we denote by (k, m, n).

Analogously, in the condition $u'_t(0, x, y) = \psi(x, y)$ we replace

$$u'_t(0, m\delta, n\delta) \quad \text{by} \quad \frac{\bar{u}(\Delta, m\delta, n\delta) - \bar{u}(0, m\delta, n\delta)}{\delta},$$

and in the condition $u(0, x, y) = \phi(x, y)$ we replace

$$u(0, m\delta, n\delta) \quad \text{by} \quad \bar{u}(0, m\delta, n\delta).$$

Then the initial conditions determine the values of $\bar{u}(0, m\delta, n\delta)$ and $\bar{u}(\Delta, m\delta, n\delta)$ at all net points for which the corresponding points $(0, m\delta, n\delta)$ lie in G.

Now writing out the difference equations $(1, m, n)$, we find the values $\bar{u}(2\Delta, m\delta, n\delta)$ at all points $(2\Delta, m\delta, n\delta)$ which are the vertices A' of pyramids of the form shown in fig. 8. Here it is assumed that all points $[0, (m \pm 1)\delta, (n \pm 1)\delta]$ lie in the interior of the square G, i.e. that

$$m_1 + 1 < m < m_2 - 1, \quad n_1 + 1 < n < n_2 - 1.$$

Next, writing out the equations $(2, m, n)$, we find the values of $\bar{u}$ at the points $(3\Delta, m, n)$, where

$$m_1 + 2 < m < m_2 - 2, \quad n_1 + 2 < n < n_2 - 2,$$

using the already determined values of $\bar{u}$ on the planes

$$t = \Delta, \quad t = 2\Delta.$$

Continuing this procedure we find the values of $\bar{u}$ at all points $(k\Delta, m\delta, n\delta)$ lying in the interior of the pyramid with base G in the plane $t = 0$ and with sides forming an angle arc tan (Δ/δ) with this plane.

If $\Delta < \delta$ and δ is sufficiently small then it can be shown that the values of $\bar{u}(k\Delta, m\delta, n\delta)$ differ by little from the values at the points $(k\Delta, m\delta, n\delta)$ of the function $u(t, x, y)$ which is the exact solution of the Cauchy problem under investigation.

The approximate values of $u(t, x, y)$ for $t < 0$ are determined in an analogous manner.

Similar constructions allow us to find an approximate solution of the Cauchy problem for more general hyperbolic equations of the second order with an arbitrary number of independent variables (cf. R. Courant, K. Friedrichs and H. Lewy, 'Ueber die partiellen Differenzengleichungen der mathematischen Physik', *Math. Ann.* vol. C (1928), pp. 32 ff.)

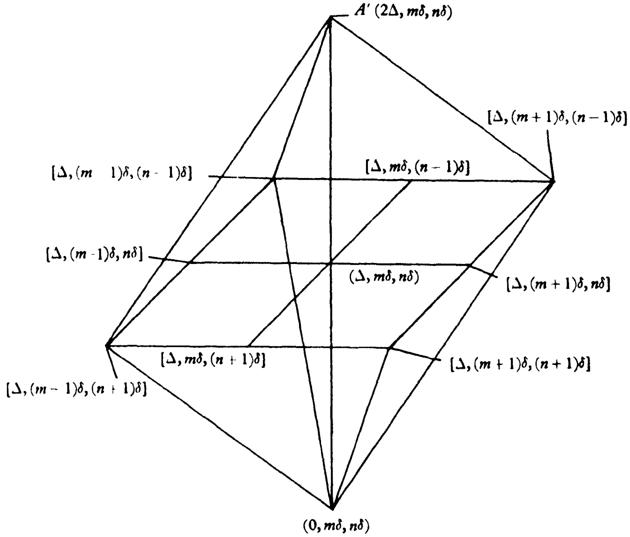
$A'\,(2\Delta, m\delta, n\delta)$
$[\Delta, (m+1)\delta, (n-1)\delta]$
$[\Delta, (m-1)\delta, (n-1)\delta]$
$[\Delta, m\delta, (n-1)\delta]$
$[\Delta, (m-1)\delta, n\delta]$
$(\Delta, m\delta, n\delta)$
$[\Delta, (m+1)\delta, n\delta]$
$[\Delta, m\delta, (n+1)\delta]$
$[\Delta, (m+1)\delta, (n+1)\delta]$
$[\Delta, (m-1)\delta, (n+1)\delta]$
$(0, m\delta, n\delta)$

Figure 8

PART II

VIBRATIONS OF BOUNDED BODIES

§ 17. Introduction

1. Part I of this chapter was devoted to the Cauchy problem. We were primarily interested in the wave equation (1, 12) which governs vibrations of homogeneous isotropic elastic bodies. The study of the function $u(t_1, x_1, \ldots, x_n)$ which characterizes these vibrations at points $(x_1, x_2, \ldots, x_n)$ for t sufficiently close to the initial moment leads to the Cauchy problem. Since the value of the solution $u(t, x_1, \ldots, x_n)$ of equation (1, 12) at the vertex $P(t, x_1, \ldots, x_n)$ of the characteristic cone is completely determined by the values

of the initial functions ϕ_0 and ϕ_1 on the base C_P of this cone, we can, when studying $u(t, x_1, \dots, x_n)$, ignore the effect of the boundary as long as C_P does not extend beyond the region in which the functions ϕ_0 and ϕ_1 are prescribed, i.e. as long as C_P does not intersect the boundary of the body. It is in this sense that we may be said to have studied in Part I vibrations of infinite or unbounded bodies.

In Part II we shall study vibrations of bodies taking into account their boundaries. Limiting ourselves again to the study of vibrations of homogeneous isotropic bodies, we are led to the problem of finding a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \tag{1, 17}$$

satisfying at $t = 0$ the *initial conditions*

$$\left.\begin{aligned} u(0, x_1, \dots, x_n) &= \phi_0(x_1, \dots, x_n), \\ u_t'(0, x_1, \dots, x_n) &= \phi_1(x_1, \dots, x_n), \end{aligned}\right\} \tag{2, 17}$$

when the point $(x_1, \dots, x_n)$ is in a given region G, and certain *boundary conditions* prescribed on the boundary of G and valid for all values of t. We shall consider only homogeneous boundary conditions of the form

$$u = 0, \tag*{$(3, 17)_1$}$$

$$\partial u / \partial n = 0, \tag*{$(3, 17)_2$}$$

$$(\partial u / \partial n) + \sigma u = 0, \tag*{$(3, 17)_3$}$$

where σ is a non-negative continuous function independent of t, given on the boundary of G, and $\partial/\partial n$ denotes differentiation in the direction of the outer normal to the boundary of G (cf. § 1).

Certain physical problems, such as the problem of vibration of non-homogeneous elastic bodies, lead to the problem of finding a solution not of equation (1, 17) but of an equation of the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p_i \frac{\partial u}{\partial x_i}\right) - qu + f, \tag{4, 17}$$

under the initial conditions (2, 17) and boundary conditions (3, 17). Here ρ, p_i, q and f are certain sufficiently smooth functions of x_k and, ordinarily,

$$\rho > \rho_0 > 0, \quad p_i > p_{i0} > 0, \quad q \geqslant 0.$$

Since equations (1, 17) and (4, 17) do not change if t is replaced by $-t$, all considerations connected with the solution of these equations for $t > 0$ are valid for $t < 0$.

The problem of finding a solution of equation (4, 17) satisfying the initial conditions (2, 17) and the boundary conditions (3, 17) is called the *mixed initial and boundary-value problem.* All of Part II of this chapter is devoted to this problem.

2. The mixed initial and boundary-value problem is not the only type of problem which can be posed for equations (4, 17) and (1, 17) in a bounded region. Practical questions often lead to other problems for these equations. A few such problems are now given for the simplest equation

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2. \tag{5, 17}$$

(1) *The Goursat problem. Find a solution of equation* (5, 17) *given its values on two characteristic segments.*

On the interval OA (cf. fig. 9) of the characteristic $t+x=0$

$$u(t, x) = \phi(x).$$

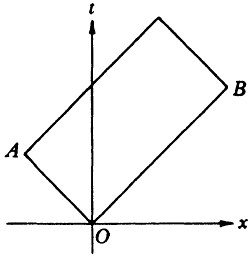

Figure 9

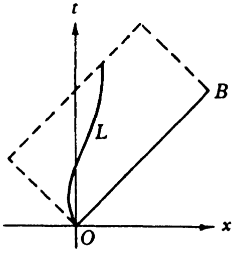

Figure 10

On the interval OB of the characteristic $t-x=0$

$$u(t, x) = \psi(x).$$

Also, for the solution to be continuous we must have

$$\phi(0) = \psi(0).$$

(2) *Find a solution of equation* (5, 17) *given its values on the interval OB of the characteristic $t=x$ and on the curve L which emanates from the origin, lies in the angle formed by the characteristics $t = \pm x$, and is such that every characteristic $t=x+C$ cuts it in one point only* (cf. fig. 10).

The reader will be able to solve these problems by himself using the solution

$$u(t, x) = f_1(t+x) + f_2(t-x)$$

of equation (5, 17) (cf. example 2, § 6).

In both cases solutions can be found in the rectangle formed by the characteristics through the end-points of the curves on which the values of u are prescribed.

(3) If the values of the function $u(t,x)$ are prescribed on two curves L and L_1 (which for the sake of simplicity we assume to be straight lines) emanating from the origin, then two essentially different cases arise: (*a*) L and L_1 lie in the interior of the angle formed by the characteristics at the origin; (*b*) L and L_1 are separated by a characteristic.

In the first case a unique solution of equation (5, 17) is determined by prescribing on L and L_1 the values of $u(t,x)$ only. In the second case it is necessary to prescribe on one of the curves L and L_1 'Cauchy data', i.e. the values of the solution and of its first derivative in the direction of the normal to this curve (cf. Goursat, *Course of Mathematical Analysis*).

3. Our subsequent considerations will, in the majority of cases, apply for any value of n. In view of the convenience in calculations and drawings, we shall conduct many of our arguments for $n=2$ or for $n=1$, giving separate formulations for other values of n only if such formulations differ essentially from those for $n=2$ or $n=1$.

Thus, assuming $n=2$, we shall consider solutions $u(t,x_1,x_2)$ of equations of the form (1, 17) and (4, 17) for $0 \leqslant t \leqslant T$, when the point (x_1,x_2) is an interior point of a region G bounded by a curve consisting of a finite number of arcs l_i with continuously turning tangent. We assume, that as we approach an end-point A of an arc l_i, the tangent approaches the tangent to the arc l_i at A.

An equivalent statement is the following. Assuming $n=2$, we shall consider solutions $u(t,x_1,x_2)$ of equations of the form (1, 17) and (4, 17) defined in the interior of a cylinder C_T with generators parallel to the t-axis and going through the boundary of a region G in the plane $t=0$ and with bases lying in the planes $t=0$ and $t=T$, respectively. In all of Part II the solutions $u(t,x_1,x_2)$ considered are assumed to satisfy equation (1, 17) or equation (4, 17) in the interior of C_T and to be twice continuously differentiable in $\overline{C_T}$, i.e. in C_T as well as on its boundary.

§ 18. Uniqueness of the mixed initial and boundary-value problem

Let $u_1(t,x_1,x_2)$ and $u_2(t,x_1,x_2)$ be two solutions of the equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}, \tag{1, 18}$$

defined in the cylinder C_T and having all the properties listed in the preceding paragraph. Let u_1 and u_2 be solutions of the same mixed initial and boundary-value problem. In other words, we assume that the two solutions satisfy the same initial conditions (2, 17) for $t=0$ and the same boundary conditions of one of the types (3, 17) on the surface of the cylinder C_T exclusive of its bases. We now wish to show that the two functions $u_1(t, x_1, x_2)$ and $u_2(t, x_1, x_2)$ coincide at all points of the cylinder C_T. The proof of this assertion is equivalent to the proof of the following theorem.

THEOREM. *The function*

$$u(t, x_1, x_2) = u_2(t, x_1, x_2) - u_1(t, x_1, x_2),$$

which satisfies equation (1, 18) *in the interior of* C_T, *which is twice continuously differentiable in* $\overline{C}_T$, *which satisfies one of the conditions* (3, 17) *on the surface of* C_T *exclusive of its bases, and which vanishes together with* u'_t *for* $t=0$, *is identically equal to zero in* C_T.

Proof. We consider the integral

$$\iiint \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}\right) dt\, dx_1\, dx_2 \tag{2, 18}$$

over the cylinder C_{t^*}, $0 < t^* \leqslant T$. Since the function u satisfies equation (1, 18), this integral must vanish. We transform this integral into an integral over the surface of the cylinder C_{t^*} in a manner analogous to that used in § 11 in proving the uniqueness of the solution of the Cauchy problem.

We get

$$\frac{1}{2}\iint_G \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1^2}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]_{t=t^*} dx_1\, dx_2$$
$$- \frac{1}{2}\iint_G \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]_{t=0} dx_1\, dx_2$$
$$- \int_0^{t^*} dt \int_l \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_1}\cos(n, x_1) + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x_2}\cos(n, x_2)\right] ds = 0.$$

As usual, l denotes the boundary of the region G and ds denotes an arc element of the boundary. The first integral is taken over the upper base of the cylinder C_{t^*}, the second over its lower base, and the third over the lateral surface of the cylinder. The latter integral can be rewritten in the form

$$\int_0^{t^*} dt \int_l \frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\, ds.$$

Thus we finally obtain

$$\frac{1}{2}\iint_G \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]_{t=t^*} dx_1 dx_2$$
$$-\frac{1}{2}\iint_G \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]_{t=0} dx_1 dx_2$$
$$-\int_0^{t^*} dt \int_l \frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\, ds = 0. \qquad (3, 18)$$

In view of the initial conditions, the second of these integrals vanishes. Now consider the third integral. It is equal to zero if $u = 0$ on the boundary of G, for then $\partial u/\partial t$ is also equal to zero on this boundary. It is also equal to zero if $\partial u/\partial n = 0$ on the boundary of G. Finally, if on the boundary

$$(\partial u/\partial n) + \sigma u = 0,$$

then our integral goes over into the integral

$$-\int_0^{t^*} dt \int_l \sigma u \frac{\partial u}{\partial t}\, ds = -\frac{1}{2}\int_l \sigma\, ds \int_0^{t^*} \frac{\partial (u^2)}{\partial t}\, dt$$
$$= -\frac{1}{2}\int_l \sigma u^2(t^*)\, ds + \frac{1}{2}\int_l \sigma u^2(0)\, ds. \qquad (4, 18)$$

The latter integral vanishes in view of the initial conditions. Hence, for all t^* between 0 and T,

$$\iint_G \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]_{t=t^*} dx_1 dx_2 = 0, \qquad (5, 18)$$

provided $u = 0$ or $\partial u/\partial n = 0$ on the boundary of G. On the other hand, if

$$(\partial u/\partial n) + \sigma u = 0$$

at all points of the boundary of G, then

$$\iint_G \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]_{t=t^*} dx_1 dx_2 + \frac{1}{2}\int_l \sigma u^2(t^*)\, ds = 0. \qquad (6, 18)$$

Since $\sigma \geqslant 0$ and $ds > 0$ (cf. § 1), the relations (5, 18) and (6, 18) imply that for each of the boundary conditions (3, 17)

$$\iint_G \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]_{t=t^*} dx_1 dx_2 = 0. \qquad (7, 18)$$

Since we are assuming that all first derivatives of u are continuous in C_T and that t^* is an arbitrary number between zero and T, the relation (7, 18) implies that in all of $\bar{C}_T$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x_1}=\frac{\partial u}{\partial x_2}=0.$$

This means that u is constant in C_T. And since $u(0,x_1,x_2)\equiv 0$, we must have

$$u(t,x_1,x_2)=0$$

in all of C_T. This completes our proof.

Using similar considerations and slightly more complicated computations, we can prove the uniqueness of the solution of the mixed initial and boundary-value problem for equation (4, 17).

We note that, apart from a constant multiplier, the integral on the left side of (7, 18) is equal to the sum of the kinetic and potential energies of a vibrating membrane at the moment $t=t^*$.

§ 19. Continuous dependence of the solution on the initial data

THEOREM. *Let there be given for $n=1$ two solutions $u_1(t,x)$ and $u_2(t,x)$ of equation* (1, 17) *in the cylinder C_T.*† *Let both solutions satisfy boundary conditions of any one of the types* (3, 17) *on the lateral surface of C_T, and for $t=0$ the initial conditions*

$$u_1(0,x)=\phi_0^{(1)}(x), \quad u'_{1t}(0,x)=\phi_1^{(1)}(x),$$
$$u_2(0,x)=\phi_0^{(2)}(x), \quad u'_{2t}(0,x)=\phi_1^{(2)}(x).$$

If the differences

$$\phi_i^{(2)}(x)-\phi_i^{(1)}(x)=\phi_i(x) \quad (i=0,1)$$

and their first derivatives are sufficiently small at all points of G, then the absolute value of the difference

$$u_2(t,x)-u_1(t,x)=u(t,x)$$

is as small as we please in all of C_T

An analogous theorem holds for solutions of equation (1, 17) in C_T for an arbitrary n. To insure this smallness of

$$u_2(t_1,x_1,\ldots,x_n)-u_1(t,x_1,\ldots,x_n)=u(t,x_1,\ldots,x_n)$$

in all of C_T in this general case, however, we must require that the functions

$$u(0,x_1,\ldots,x_n) \quad \text{and} \quad u'_t(0,x_1,\ldots,x_n)$$

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† Clearly, for $n=1$, the cylinder C_T is simply a rectangle with sides parallel to the t- and x-axes.

and all their derivatives with respect to $x_1, \ldots, x_n$ of order up to and including $[\frac{1}{2}n]+1$ differ by little from zero at all points of G. In addition to our assumption certain supplementary conditions, which hold automatically in the case $n=1$, must hold for the derivatives of order up to $[\frac{1}{2}n]$ of these differences at all points of the boundary of G in the base of C_T. In view of the complexity of the proof for the case $n>1$, we shall give the proof for the case $n=1$ only.

Proof of the theorem for $n=1$. We again consider an integral of the form (2, 18) over the cylinder C_T which now degenerates into a rectangle. As before, this integral vanishes for all t^* between zero and T. Transforming it as before, we get

$$\iint_{C_{t^*}} \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}\right) dt\,dx$$

$$=\frac{1}{2}\int_a^b \left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]_{t=t^*} dx-\frac{1}{2}\int_a^b \left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]_{t=0} dx$$

$$-\int_0^{t^*}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)_{x=b} dt+\int_0^{t^*}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)_{x=a} dt=0.$$

Whence, in view of the fact that $a<b$ and[†]

$$u(0,x)=\phi_0, \quad u_t'(0,x)=\phi_1, \quad \left.\frac{\partial u}{\partial x}\right|_{x=b}=\frac{\partial u}{\partial n}, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a}=-\frac{\partial u}{\partial n},$$

we get

$$\frac{1}{2}\int_a^b \left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]_{t=t^*} dx$$

$$=\frac{1}{2}\int_a^b [\phi_1^2(x)+\phi_0'^2(x)]\,dx+\int_0^{t^*}\left.\frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\right|_{x=a} dt+\int_0^{t^*}\left.\frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\right|_{x=b} dt. \tag{1, 19}$$

If $u(t,a)=0$ or $\partial u(t,a)/\partial n=0$, then

$$\int_0^{t^*}\left.\frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\right|_{x=a} dt=0.$$

On the other hand, if for $x=a$, the boundary condition

$$(\partial u/\partial n)+\sigma_a u=0$$

[†] We recall that $\partial/\partial n$ always denotes differentiation in the direction of the outer normal.

holds, then

$$\int_0^{t^*} \frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\bigg|_{x=a} dt = -\sigma_a \int_0^{t^*} u\frac{\partial u}{\partial t}\,dt = -\frac{\sigma_a u^2(t^*,a)}{2} + \frac{\sigma_a u^2(0,a)}{2}.$$

Similar equalities can be written down at $x=b$ if at $x=b$ one of the following conditions is imposed:

$$u(t,b)=0, \quad \partial u(t,b)/\partial n = 0.$$

Thus, rejecting, if necessary, negative terms on the right side of (1, 19), we get for each type of boundary conditions (3, 17)

$$\int_a^b \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]_{t=t^*} dx \leqslant \int_a^b [\phi_1^2(x) + \phi_0'^2(x)]\,dx + \sigma_a \phi_0^2(a) + \sigma_b \phi_0^2(b). \quad (2, 19)\dagger$$

The assumed smallness of the right side implies the smallness of the left side. Denoting by ϵ^2 the value of the right side of (2, 19), we find that for every t^* between zero and t and for every x $(a \leqslant x \leqslant b)$,

$$\int_a^x \left(\frac{\partial u}{\partial x}\right)^2_{t=t^*} dx \leqslant \epsilon^2, \quad (3, 19)_1$$

$$\int_a^x \left(\frac{\partial u}{\partial t}\right)^2_{t=t^*} dx \leqslant \epsilon^2. \quad (3, 19)_2$$

Applying Schwarz's inequality to the inequality $(3, 19)_1$, we find that

$$|\, u(t^*,x) - u(t^*,a)\,| \leqslant \int_a^x \left|\frac{\partial u}{\partial x}\right| dx$$

$$= \int_a^x 1 \left|\frac{\partial u}{\partial x}\right| dx \leqslant \left[\int_a^x dx \int_a^x \left(\frac{\partial u}{\partial x}\right)^2 dx\right]^{\frac{1}{2}} \leqslant (b-a)^{\frac{1}{2}}\,\epsilon. \quad (4, 19)$$

From the inequality $(3, 19)_2$ we obtain in a similar manner the inequality

$$\left|\frac{\partial}{\partial t}\int_a^b u\,dx\right| = \left|\int_a^b \frac{\partial u}{\partial t}\,dx\right| \leqslant \int_a^b \left|\frac{\partial u}{\partial t}\right| dx \leqslant (b-a)^{\frac{1}{2}}\,\epsilon. \quad (5, 19)$$

This inequality implies that

$$\left|\int_a^b u(t^*,x)\,dx\right| \leqslant t^*\epsilon(b-a)^{\frac{1}{2}} + \max|\,\phi_0\,|\,(b-a). \quad (6, 19)$$

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† For boundary conditions $u=0$ or $\partial u/\partial n = 0$ this inequality becomes an equality expressing the law of conservation of energy.

Integrating the inequality (4, 19) with respect to x from $x=a$ to $x=b$, we find that

$$\left|\int_a^b u(t^*,x)\,dx-(b-a)\,u(t^*,a)\right|\leqslant\epsilon(b-a)^{\frac{3}{2}},\qquad(7,19)$$

and, estimating the value of the integral on the left side of (7, 19) by means of (6, 19), we see that the values of $u(t^*,a)$ are as small as we please for all t^* in the interval $(0,T)$ for sufficiently small ϵ and $\max|\phi_0|$. Making use again of the inequality (4, 19) we now obtain the required result: $u(t^*,x)$ is small in all of the rectangle C_T.

Remark. If at one end of the interval $[a,b]$, at the end $x=a$, say, the condition

$$u=0\quad\text{for all}\quad t\geqslant 0$$

were given, then the smallness of $|u(t,x)|$ in the interior of C_T would follow directly from (4, 19).

If at one end of the interval $[a,b]$, at the end $x=a$, say, the condition

$$(\partial u/\partial n)+\sigma_a u=0\quad\text{for}\quad\sigma_a>0$$

were given, then (1, 19) would imply that

$$\int_a^b\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]_{t=t^*}dx+\sigma_a u^2(t^*,a)$$

$$\leqslant\int_a^b[\phi_1^2(x)+\phi_0'^2(x)]\,dx+\sigma_a\phi_0^2(a)+\sigma_b\phi_0^2(b)\leqslant\epsilon^2.$$

Whence

$$\sigma_a u^2(t^*,a)\leqslant\epsilon^2,$$

and the smallness of $u(t^*,x)$ in C_T would again follow directly from (4, 19) in view of the smallness of $u(t^*,a)$.

Remark 2. If $n>1$, then the assumed uniform smallness of $|u(0,x_1,\ldots,x_n)|$ and $|u_t'(0,x_1,\ldots,x_n)|$ would, just as in the case $n=1$, imply the smallness of

$$\int_G\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial u}{\partial x_1}\right)^2+\ldots+\left(\frac{\partial u}{\partial x_n}\right)^2\right]_{t=t^*}dx_1\ldots dx_n.\qquad(8,19)$$

The smallness of this integral for all t^* between zero and T and the smallness of $|u(0,x_1,\ldots,x_n)|$ would not suffice to insure the smallness of $|u|$ in C_T, however. It is possible to give an example of a function u for which this integral is small for all values t^* considered which, nevertheless, assumes very large values at some points of C_T, regardless of the smallness of $|u(0,x_1,\ldots,x_n)|$. To guarantee the smallness of $|u|$ in C_T it would be sufficient to require that together with the integral (8, 19) all integrals of the form (8, 19) with u replaced by all possible derivatives of the form

$$\frac{\partial^k u}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_n^{k_n}} \quad \text{for} \quad k \leqslant [\tfrac{1}{2}n]$$

be small for $t = t^*$, and that the values of $|u(0, x_1, \dots, x_n)|$ be uniformly small. It is under these assumptions that the smallness of $|u|$ in C_T can be proved for ϕ_0 and ϕ_1 and their derivatives with respect to $x_1, \dots, x_n$ of order up to $[\frac{1}{2}n] + 1$ small and absolute value, provided certain supplementary conditions hold for the values of ϕ_0 and ϕ_1 on the boundary of G.†

Remark 3. If in formulating the theorem proved above we restrict ourselves to boundary conditions of the form $u = 0$ and $\partial u/\partial n = 0$ at each of the end-points of the interval $[a, b]$, then the requirement of smallness of $|u(0, x)|$ and $|u_t'(0, x)|$ may be replaced by the requirement that $|u(0, a)|$, $|u(0, b)|$, and

$$\int_a^b \left(\frac{\partial u}{\partial t}\right)_{t=0}^2 dx, \quad \int_a^b \left(\frac{\partial u}{\partial x}\right)_{t=0}^2 dx$$

be small, for in this case only the above integrals appear on the right side of (2, 19).

§ 20. The Fourier method for the equation of a vibrating string

1. Many mixed initial and boundary-value problems can be solved by applying the so-called Fourier method. In this section we demonstrate the use of the Fourier method by means of an example. In the next section we shall show how this method can be applied to obtain the solution of a mixed initial and boundary-value problem for a linear second-order equation in two independent variables.

Let it be required to find a solution of the equation

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2, \tag{1, 20}$$

satisfying the initial conditions

$$\left.\begin{aligned} u(0, x) &= \phi_0(x) \\ u_t'(0, x) &= \phi_1(x) \end{aligned}\right\} \quad (0 \leqslant x \leqslant l), \tag{2, 20}$$

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† See S. L. Sobolev, *C.R. Acad. Sci. U.R.S.S.* vol. I (x), no. 7 (1936), p. 279.

and the boundary conditions

$$u(t,0)=u(t,l)=0 \quad (t>0). \tag{3, 20}$$

We first try to find a non-trivial (not identically equal to zero) solution of equation (1, 20) of the form

$$u(t,x)=T(t)\,X(x), \tag{4, 20}$$

satisfying the boundary conditions (3, 20). We assume that $T(t)$ is a function of t only and $X(x)$ a function of x only. Substituting the right side of (4, 20) in place of u in (1, 20), we get

$$XT''=X''T \quad \text{or} \quad T''/T=X''/X. \tag{5, 20}$$

The left side of the latter equality does not depend on x, and its right side does not depend on t. Consequently, both expressions T''/T and X''/X are independent of x and t and so must be equal to a common constant. Let λ be this constant. Then (5, 20) implies that

$$T''-\lambda T=0, \tag{6, 20}$$

$$X''-\lambda X=0. \tag{7, 20}$$

Thus equation (5, 20) yields two equations of which one contains only functions of t and the other only functions of x. In such cases we say that *the variables have been separated.*

In order to obtain a non-trivial solution $u(t,x)$ of the form (4, 20), satisfying the boundary conditions (3, 20), it is necessary to find a non-trivial (i.e. not identically equal to zero) solution of equation (7, 20) satisfying the boundary conditions

$$X(0)=X(l)=0. \tag{8, 20}$$

The form of the general real solution of (7, 20) depends in a fundamental manner on whether

$$\lambda>0, \quad \lambda=0 \quad \text{or} \quad \lambda<0.$$

I. *The case* $\lambda>0$. Here the general solution of equation (7, 20) can be written in the form

$$X(x)=C_1\exp\{\lambda^{\frac{1}{2}}x\}+C_2\exp\{-\lambda^{\frac{1}{2}}x\}.$$

For the boundary conditions (8, 20) to hold, we must have

$$C_1+C_2=0 \quad \text{and} \quad C_1\exp\{\lambda^{\frac{1}{2}}l\}+C_2\exp\{-\lambda^{\frac{1}{2}}l\}=0.$$

Consequently, we must have

$$C_1\exp\{\lambda^{\frac{1}{2}}l\}=C_2\exp\{-\lambda^{\frac{1}{2}}l\}.$$

The latter inequality can be satisfied only if $C_1 = 0$. But this implies that $C_2 = 0$. Thus, in this case, we obtain a trivial solution of equation (7, 20).

II. *The case* $\lambda = 0$. Here the general solution of equation (7, 20) is of the form

$$X(x) = C_1 + C_2 x.$$

For $X(0) = 0$ to hold, C_1 must be zero. Then the condition $X(l) = 0$ takes the form $C_2 l = 0$, i.e. $C_2 = 0$. Consequently, just as in case I, we see that here, too, the only solution of equation (7, 20) which satisfies both boundary conditions (8, 20) is the trivial solution.

III. *The case* $\lambda < 0$. Here the general solution has the form

$$X(x) = C_1 \cos(-\lambda)^{\frac{1}{2}} x + C_2 \sin(-\lambda)^{\frac{1}{2}} x.$$

For the boundary condition $X(0) = 0$ to hold, we must have

$$C_1 = 0.$$

Then the condition $X(l) = 0$ becomes

$$C_2 \sin(-\lambda)^{\frac{1}{2}} l = 0 \quad \text{or} \quad \sin(-\lambda)^{\frac{1}{2}} l = 0,$$

for $C_2 = 0$ would again lead to a trivial solution. The equation

$$\sin(-\lambda)^{\frac{1}{2}} l = 0$$

is satisfied if and only if

$$(-\lambda)^{\frac{1}{2}} l = k\pi, \quad \text{e.g.} \quad \lambda = -(k^2\pi^2/l^2),$$

where k is any integer or zero. Since we are assuming that $\lambda < 0$, k cannot be equal to zero. The value of λ is independent of the sign of k. Consequently, all values of λ for which equation (7, 20) has a non-trivial solution satisfying the boundary conditions (8, 20) are given by the formula

$$\lambda_k = -(k^2\pi^2/l^2) \quad \text{where} \quad k = 1, 2, \ldots. \tag{9, 20}$$

The problem of finding non-trivial solutions of equation (7, 20) satisfying the boundary conditions (8, 20) is a special case of the so-called 'eigenvalue problem' or the 'Sturm-Liouville problem'. The values of $-\lambda$ for which our problem has non-trivial solutions are called *eigenvalues*, and the non-trivial solutions are called *eigenfunctions* corresponding to the given eigenvalue. In our case the eigenfunction corresponding to the eigenvalue $k^2\pi^2/l^2$ is

$$X_k(x) = C_k \sin(k\pi/l)\, x.$$

In view of the homogeneity of equation (7, 20), the eigenfunctions are determined to within a constant multiplicative factor C_k. By suitably choosing this factor we can make the function $X_k(x)$ satisfy a certain supplementary condition, or, as we say, *normalize* the eigenfunction. We will find it convenient to normalize our eigenfunctions so that

$$\int_0^l X_k^2(x)\,dx = 1.$$

For this we must have $C_k = (\frac{1}{2}l)^{\frac{1}{2}}$.

For the rest of this section the symbol $X_k(x)$ shall mean

$$X_k(x) = (\tfrac{1}{2}l)^{\frac{1}{2}} \sin(k\pi/l)\,x.$$

We now turn to the solution of the mixed initial and boundary-value problem formulated in the beginning of this section. Replacing λ in (6, 20) by its value λ_k given by formula (9, 20), we get

$$T'' + (k^2\pi^2/l^2)\,T = 0.$$

Whence $T_k(t) = A_k \cos(k\pi/l)\,t + B_k \sin(k\pi/l)\,t,$

where A_k and B_k are arbitrary constants.

All functions

$$u_k(t,x) = X_k(x)\,T_k(t) = (\tfrac{1}{2}l)^{\frac{1}{2}} \sin\frac{k\pi}{l}x\left(A_k\cos\frac{k\pi}{l}t + B_k\sin\frac{k\pi}{l}t\right)$$

satisfy equation (1, 20) and the boundary conditions (3, 20) for arbitrary A_k and B_k. We now try to determine these constants so that the infinite series

$$u(t,x) = \sum_{k=1}^{\infty} X_k(x)\left[A_k\cos\frac{k\pi}{l}t + B_k\sin\frac{k\pi}{l}t\right] \qquad (10, 20)$$

satisfies equation (1, 20), the boundary conditions (3, 20), and the initial conditions (2, 20). We start with the initial conditions. First

$$u(0,x) = \sum_{k=1}^{\infty} A_k X_k(x) = \phi_0(x) \qquad (11, 20)$$

must hold. In addition, assuming that the series is termwise differentiable, we must have

$$u_t'(0,x) = \sum_{k=1}^{\infty} \frac{k\pi}{l} B_k X_k(x) = \phi_1(x). \qquad (12, 20)$$

Let us assume that the functions $\phi_0(x)$ and $\phi_1(x)$ can be represented in the interval $(0, l)$ by absolutely and uniformly convergent series in $\sin (k\pi/l)\, x$. We know from the theory of trigonometric series that this can always be done if the functions $\phi_0(x)$ and $\phi_1(x)$ are continuous together with their first derivatives and if these functions vanish at the end-points of the interval $(0, l)$. Let these conditions be satisfied. Then, in view of the fact that

$$|\sin (k\pi/l)\, t\,| \leqslant 1, \quad |\cos (k\pi/l)\, t\,| \leqslant 1,$$

the series (10, 20) converges absolutely and uniformly for $0 \leqslant x \leqslant l$ and for arbitrary t. This means that the function $u(t, x)$ defined by the series (10, 20) is continuous and that it satisfies the first initial condition (2, 20) as well as the boundary conditions (3, 20). This, however, does not justify the conclusion that this function satisfies the second initial condition (2, 20) and equation (1, 20). Such conclusion would be valid if the series (10, 20) were differentiable twice with respect to x and twice with respect to t. As is well known, termwise differentiation is allowed if the formal result of this operation yields series which converge uniformly in C_T. The latter condition will certainly be satisfied for all values of T including $T = \infty$ if the functions ϕ_0 and ϕ_1 are four times continuously differentiable in $[0, l]$ and if the values of the functions and of their derivatives of order two vanish at the end-points of the interval $[0, l]$. Then†

$$A_k = O(1/n^4) \quad \text{and} \quad B_k = O(1/n^4).$$

2. *Remark.* In practice we usually do not worry about the possibility of differentiating the series (10, 20) termwise twice with respect to x and t. We are satisfied if the functions ϕ_0 and ϕ_1 are continuous together with their first derivatives and if the functions themselves vanish at the end-points of the interval $(0, l)$. As we saw, this insures the uniform and absolute convergence of the series (10, 20) at all points of the rectangle C_T. If, for the given functions ϕ_0 and ϕ_1, there exists in C_T a twice continuously differentiable solution $u(t, x)$ of the problem considered, then the sequence of partial sums $S_k(t, x)$ of the series (10, 20) converges uniformly to this solution in all of C_T. In fact, we know from the theory of trigonometric series that the Fourier series of every square integrable function converges to this function in the mean. Consequently, it follows from the very construction of the series

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† By $O\,[\phi(n)]$ we denote any function $\psi(n)$ for which the quotient $\psi(n)/\phi(n)$ remains bounded as $n \to \infty$.

(10, 20) that

$$\int_0^l [S'_{kx}(0,x)-\phi'_0(x)]^2\,dx \to 0 \quad \text{and} \quad \int_0^l [S'_{kt}(0,x)-\phi_1(x)]^2\,dx \to 0$$

for $k \to \infty$.

It now follows on the basis of remark 3 in § 19 that

$$S_k(t,x) \underset{k\to\infty}{\to} u(t,x)$$

uniformly in all of $\bar{C}_T$.

It can happen, however, that, notwithstanding the assumptions made about ϕ_0 and ϕ_1 in $\bar{C}_T$, no twice continuously differentiable solution of our problem exists. Then the sum of the series (10, 20) is a generalized solution of this problem. In fact, for an arbitrary subscript k the partial sum $S_k(t,x)$ of the series (10, 20) satisfies equation (1, 20) and the boundary conditions (3, 20), and for k sufficiently large this partial sum satisfies the initial conditions (2, 20) to within an error which is arbitrarily small in all of the interval $(0, l)$. On the other hand the series (10, 20) differs from $S_k(t,x)$ by a uniformly small amount in all of $\bar{C}_T$.

3. Each of the functions

$$u_k(t,x) = X_k(x)\,T_k(t) = \sin\frac{k\pi}{l}x\left(A_k\cos\frac{k\pi}{l}t + B_k\sin\frac{k\pi}{l}t\right)$$

$$= C_k\sin\frac{k\pi}{l}x\sin\frac{k\pi}{l}(t+t_0) \quad (k=1,2,3,\dots)$$

represents a so-called 'eigenvibration' of a string fastened at its ends. When vibrating according to the 'eigenvibrations' corresponding to $k=1$, the string emits the basic, lowest tone. For larger values of k we get the 'overtones'. When vibrating according to the law

$$u(t,x) = \sum_{k=1}^{n} C_k\sin\frac{k\pi}{l}x\sin\frac{k\pi}{l}(t+t_0),$$

the string gives off simultaneously tones of different frequencies corresponding to the individual terms of this sum.

§ 21. The general Fourier method (introductory considerations)

The Fourier method (also known as the method of separation of variables) for solving mixed initial and boundary-value problems is applicable to a certain special class of linear second-order equations only, although the problem itself can be solved for a much more extensive class of equations.

In this section we give a non-rigorous presentation of the method. The first to give a rigorous justification of the Fourier method was V. A. Steklov. In the following section we give two different justifications of the Fourier method, one of them involving variational principles and the other reduction to integral equations.

Let us consider the hyperbolic equation

$$A(t)\frac{\partial^2 u}{\partial t^2}+C(x)\frac{\partial^2 u}{\partial x^2}+D(t)\frac{\partial u}{\partial t}+E(x)\frac{\partial u}{\partial x}+[F_1(t)+F_2(x)]\,u=0, \tag{1,21}$$

where the coefficients A, C, D, E, F_1, F_2 are sufficiently smooth functions with

$$A(t)>a_0>0, \quad C(x)<c_0<0; \qquad a_0, c_0=\text{const.}$$

We assume that some of the coefficients are functions of x only, and others of t only; that not because such assumption is called for by the nature of the problem before us but because we wish to apply the Fourier method.

Let it be required to find a twice continuously differentiable solution of equation (1, 21), satisfying the initial conditions

$$u(0,x)=\phi_0(x), \quad u_t'(0,x)=\phi_1(x), \tag{2,21}$$

and the boundary conditions

$$\left.\begin{aligned} A_0u(t,0)+B_0u_x'(t,0)=0,\\ A_1u(t,l)+B_1u_x'(t,l)=0.\end{aligned}\right\} \tag{3,21}$$

As in the example in § 20, we first try to find a non-trivial solution of equation (1, 21) in the form of a product

$$u(t,x)=T(t)\,X(x) \tag{4,21}$$

satisfying the boundary conditions (3, 21). We ignore as yet the matter of initial conditions.

If such a solution exists, then, substituting it in (1, 21), we obtain an equation which must necessarily be satisfied by the functions $X(x)$ and $T(t)$:

$$A(t)\,T''X+C(x)\,TX''+D(t)\,T'X+E(x)\,TX'+[F_1(t)+F_2(x)]\,TX=0.$$

Since $X(x)$ and $T(t)$ cannot both vanish identically, we may write instead

$$A(t)\frac{T''(t)}{T(t)}+D(t)\frac{T'(t)}{T(t)}+F_1(t)=-C(x)\frac{X''(x)}{X(x)}-E(x)\frac{X'(x)}{X(x)}-F_2(x). \tag{5, 21}$$

The left side of (5, 21) is independent of x, and the right side of t. Consequently.

$$A\frac{T''}{T}+D\frac{T'}{T}+F_1(t)=-C\frac{X''}{X}-E\frac{X'}{X}-F_2(x)=-\lambda,$$

where λ is some constant.

These equalities represent a pair of ordinary differential equations for the determination of the functions $T(t)$ and $X(x)$, respectively. Multiplying the first of these equations by $T(t)$ and the second by $X(x)$, we can write them in the form

$$A(t)\,T''+D(t)\,T'+F_1(t)\,T+\lambda T=0, \qquad (6,21)\dagger$$

$$C(x)\,X''+E(x)\,X'+F_2(x)\,X-\lambda X=0. \qquad (7,21)\dagger$$

Since $T(t)\not\equiv 0$, for the function (4, 21) to satisfy the boundary conditions (3,21) the following conditions must necessarily hold:

$$\left.\begin{aligned}A_0X(0)+B_0X'(0)=0,\\ A_1X(l)+B_1X'(l)=0.\end{aligned}\right\} \tag{8, 21}$$

The problem of finding non-trivial solutions of equation (7, 21) satisfying the conditions (8, 21) is known as the *eigenvalue problem.* This problem does not have a non-trivial solution for every value of λ. The values of λ for which there exist non-trivial solutions are called the *eigenvalues* of this problem and the non-trivial solutions themselves are called *eigenfunctions* corresponding to the given eigenvalue. The totality of eigenvalues of a given problem is called its *spectrum.*

In the next section we show that the eigenvalues of our problem represent an infinite sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots.$$

To every eigenvalue λ_k there corresponds an eigenfunction $X_k(t)$ which, in view of the homogeneity of equation (7, 21) and the con-

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† This equality is obvious at those points at which $X(x)\neq 0$ and $T(t)\neq 0$. At points at which one or both functions vanish, the correctness of the equality follows from a continuity argument.

ditions (8, 21), is determined to within an arbitrary multiplicative factor. We choose this factor so that

$$\int_0^l \rho(x)\,X_k^2(x)\,dx = 1, \tag{9, 21}$$

where $\rho(x)$ is a certain as yet undefined function fixed for a given equation.

We show in the sequel that the eigenfunctions corresponding to different eigenvalues are *orthogonal with respect to the weight function* ρ, i.e. they satisfy the equalities

$$\int_0^l \rho X_k(x)\,X_l(x)\,dx = 0 \quad \text{for} \quad k \neq l. \tag{10, 21}$$

We solve equation (6, 21) for every eigenvalue λ_k. The general solution of equation (6, 21) for $\lambda = \lambda_k$ (we denote it by $T_k(t)$) represents an arbitrary linear combination of any two linearly independent particular solutions $T_k^*(t)$ and $T_k^{**}(t)$

$$T_k(t) = C_1 T_k^*(t) + C_2 T_k^{**}(t).$$

We choose $T_k^*(t)$ and $T_k^{**}(t)$ so that they satisfy, for $t = 0$, the initial conditions

$$\left.\begin{array}{ll} T_k^*(0) = 1, & T_k^{*\prime}(0) = 0, \\ T_k^{**}(0) = 0, & T_k^{**\prime}(0) = 1, \end{array}\right\} \tag{11, 21}$$

and we put

$$u_k(t, x) = T_k(t)\,X_k(x).$$

For an arbitrary value of the subscript k the functions $u_k(t, x)$ satisfy equation (1, 21) and the boundary conditions (3, 21).

To satisfy the initial conditions (2, 21), we construct the series

$$u(t, x) = \sum_{k=1}^{\infty} X_k(x)\,[A_k T_k^*(t) + B_k T_k^{**}(t)]. \tag{12, 21}$$

If this series as well as the series obtained from it by means of differentiating it termwise twice with respect to t and twice with respect to x converge uniformly, then its sum will obviously satisfy equation (1, 21) and the boundary conditions (3, 21). Then for the initial conditions (2, 21) to be satisfied, we must have

$$u(0, x) = \sum_{k=1}^{\infty} A_k X_k(x) = \phi_0(x), \tag{13, 21}$$

$$u_t'(0, x) = \sum_{k=1}^{\infty} B_k X_k(x) = \phi_1(x). \tag{14, 21}$$

Assuming that the series (13, 21) and (14, 21) converge uniformly, we can determine the coefficients A_m and B_m by multiplying both sides of the equalities (13, 21) and (14, 21) by $\rho X_m(x)$ and integrating with respect to x from 0 to l. In view of (9, 21) and (10, 21), we then get

$$A_m = \int_0^l \rho(x)\,\phi_0(x)\,X_m(x)\,dx,$$

$$B_m = \int_0^l \rho(x)\,\phi_1(x)\,X_m(x)\,dx.$$

Substituting these values for the coefficients A_k and B_k in the series (12, 21) we now obtain a solution of our problem provided that the series (12, 21), (13, 21) and (14, 21) converge rapidly enough for the series obtained from the series (12, 21) by means of differentiating it termwise twice with respect to t and twice with respect to x to converge uniformly.

§22. General properties of eigenfunctions and eigenvalues

1. To investigate the properties of eigenfunctions and eigenvalues we first show that the equation

$$C(x)\,X''(x) + E(x)\,X'(x) + F_2(x)\,X(x) - \lambda X(x) = 0 \qquad (7, 21)$$

of the preceding section can be reduced to the form

$$[p(x)\,X'(x)]' - q(x)\,X(x) + \lambda\rho X(x) = 0 \qquad (1, 22)$$

by multiplying it by a suitably chosen function of x.

We assume in the sequel that

$$C(x) < c_0 < 0, \quad c_0 = \text{const.}$$

Multiplying (7, 21) by $\rho(x)$, we get

$$\rho C X'' + \rho E X' + \rho F_2 x - \lambda\rho X = 0.$$

For the sum of the first two terms to be of the form

$$[p(x)\,X']'$$

we must have
$$(\rho C)' = \rho E.$$

A special solution of this differential equation is

$$\rho(x) = \exp\left\{\int_{x_0}^{x} \frac{E - C'}{C}\,dx\right\} > 0.$$

Putting
$$\rho C = -p \quad \text{and} \quad \rho F_2 = +q,$$

we can now write our expression in the form (1, 22).

Our assumptions imply that

$$p(x) > p_0, \quad \rho(x) > \rho_0; \qquad p_0, \rho_0 \text{ const.}$$

We assume that $p(x)$, $p'(x)$, $q(x)$, $\rho(x)$ are continuous in $0 \leqslant x \leqslant l$.

2. We now turn to the consideration of the eigenvalue problem which consists in finding a non-trivial solution of equation (1, 22) satisfying the conditions

$$\left.\begin{aligned} A_0 X(0) + B_0 X'(0) = 0,\\ A_1 X(l) + B_1 X'(l) = 0. \end{aligned}\right\} \tag{2, 22}$$

THEOREM 1. *If $X_1(x)$ and $X_2(x)$ are eigenfunctions corresponding to the same eigenvalue λ, then, for some constant c,*

$$X_1(x) = cX_2(x).$$

Proof. We first assume that $B_0 \neq 0$ and we put†

$$X_1(0)/X_2(0) = c.$$

Since both functions must satisfy the conditions (2, 22), it follows that

$$X_1'(0)/X_2'(0) = c.$$

We note that the function $X_1(x) = cX_2(x)$ satisfies the initial conditions $X_1(0) = cX_2(0)$ and $X_1'(0) = cX_2(0)$. In view of the uniqueness of the solution of a differential equation with initial conditions, this is the only function with this property. Consequently,

$$X_1(x) = cX_2(x).$$

If $B_0 = 0$, then $X_1(0) = X_2(0) = 0$. We can then demonstrate the correctness of Theorem 1 by putting $c = X_1'(0)/X_2'(0)$.

We assume in the sequel that the eigenfunctions are normalized with respect to the weight function ρ, i.e. that

$$\int_0^l \rho(x)\,[X(x)]^2\,dx = 1. \tag{3, 22}$$

$X(x)$ is obtained from an arbitrary eigenfunction $\tilde{X}(x)$ by multiplying the latter by the number

$$\left[\int_0^l \rho(x)\,[\tilde{X}(x)]^2\,dx\right]^{-\frac{1}{2}}.$$

It is clear that the normalized eigenfunction corresponding to a given eigenvalue is determined to within sign.

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† If $B_0 \neq 0$, then neither $X_1(0)$ nor $X_2(0)$ vanishes. Otherwise, (2, 22) would imply the vanishing of $X_1'(0)$ or $X_2'(0)$ so that either $X_1(x) \equiv 0$ or $X_2(x) \equiv 0$.

THEOREM 2. *Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function $\rho(x)$. This means that if $\lambda_1 \neq \lambda_2$ and $X_i(x)$ is the eigenfunction associated with the eigenvalue λ_i $(i=1,2)$, then*

$$\int_0^l \rho(x) X_1(x) X_2(x)\, dx = 0. \tag{4,22}$$

Proof. We write the identities

$$(pX_1')' - qX_1 + \lambda_1 \rho X_1 = 0,$$

$$(pX_2')' - qX_2 + \lambda_2 \rho X_2 = 0.$$

We multiply the first of these identities by X_2 and the second by X_1. Subtracting one of the resulting expressions from the other we obtain the identity

$$[pX_1']' X_2 - [pX_2']' X_1 + (\lambda_1 - \lambda_2) \rho X_1 X_2 = 0.$$

We now integrate this identity from zero to l.

$$\int_0^l \{[pX_1']' X_2 - [pX_2']' X_1\}\, dx$$
$$= pX_1' X_2 \Big|_0^l - pX_2' X_1 \Big|_0^l - \int_0^l pX_1' X_2'\, dx + \int_0^l pX_2' X_1'\, dX.$$

Since the integrals cancel out and the remaining terms vanish in view of the initial conditions, it follows immediately that

$$(\lambda_1 - \lambda_2) \int_0^l \rho X_1 X_2\, dx = 0.$$

This implies (4, 22) (because $\lambda_1 \neq \lambda_2$).

3. To simplify exposition we limit ourselves to the consideration of boundary conditions of the form

$$X(0) = 0, \quad X(l) = 0. \tag{5,22}$$

In this section we reduce the problem of finding eigenvalues to that of finding a minimum of a certain functional. We choose this functional so that equation (1, 22) is the Lagrange-Euler equation for it.†

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† R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. I (1953), Interscience, New York–London. Also, M. A. Lavrentiev and L. A. Liusternik, *Calculus of Variations*, vol. I, part 2, ONTI (1935) (in Russian).

We consider two quadratic functionals of the function $X(x)$

$$G(X)=\int_0^l (pX'^2+qX^2)\,dx,$$

$$H(X)=\int_0^l \rho X^2\,dx.$$

The functionals

$$G(X_1,X_2)=\int_0^l (pX_1'X_2'+qX_1X_2)\,dx,$$

$$H(X_1,X_2)=\int_0^l \rho X_1X_2\,dx$$

are referred to as *bilinear functionals* corresponding to the given quadratic functionals. We now prove

THEOREM 3. *If λ is an eigenvalue of the eigenvalue problem under consideration, and if X_λ is the normalized eigenfunction corresponding to λ, then*

$$G(X_\lambda)=\lambda.$$

Proof. Note that

$$\int_0^l p(X_\lambda')^2\,dx=\int_0^l pX_\lambda'\,dX_\lambda=pX_\lambda'X_\lambda\Big|_0^l-\int_0^l (pX_\lambda')'X_\lambda\,dx,$$

and that the first term on the right vanishes in view of the conditions (5, 22). Now, the function $X(x)$ satisfies equation (1, 22) as well as the condition of normalization (3, 22) (which can now be written as $H(X_\lambda)=1$). Consequently,

$$G(X_\lambda)=-\int_0^l [(pX_\lambda')'-qX_\lambda]\,X_\lambda\,dx=\lambda\int_0^l \rho X_\lambda X_\lambda\,dx=\lambda.$$

THEOREM 4. *The ratio $G(X)/H(X)$ is bounded from below and, consequently, has a greatest lower bound.*

In fact

$$G(X)=\int_0^l (pX'^2+qX^2)\,dx\geqslant\int_0^l qX^2\,dx$$

$$=\int_0^l \frac{q}{\rho}\rho X^2\,dx\geqslant \min_{0\leqslant x\leqslant l}\frac{q(x)}{\rho(x)}H(X).$$

Consider all functions X satisfying the condition $H(X)=1$. For such functions the set of values of the functional $G(X)$ is bounded from below. And, since $H(X_\lambda)=1$ implies $\lambda=G(X_\lambda)$ for every eigenvalue λ, we obtain the following important result: *The set of eigenvalues of our problem is bounded from below.*

We now consider the problem of finding the minimum of the functional $G(X)$ with $H(X)=1$. The class of admissible functions shall be the class of functions $X(x)$ which are twice continuously differentiable on the interval $0\leqslant x\leqslant l$ and which satisfy the conditions (5, 22). *We assume that this minimum is achieved for some function in the class of admissible functions.*† Then, as is known from the calculus of variations, the function for which the minimum is achieved must, for some value of λ, satisfy the Lagrange-Euler equation for the functional

$$G(X)-\lambda H(X)=\int_0^l [pX'^2+qX^2-\lambda\rho X^2]\,dx$$
$$=\int_0^l F(x, X, X')\,dx,$$

i.e. an equation of the form

$$\frac{\partial F}{\partial X}-\frac{d}{dx}\frac{\partial F}{\partial X'}=0,$$

which, in our case, coincides with equation (1, 22). The boundary conditions of the eigenvalue problem and of the variational problem considered also coincide. Consequently, the function $X_1(x)$ which minimizes $G(X)$ with $H(X)=1$ is an eigenfunction of the original eigenvalue problem. Since, by Theorem 3, $G(X_\lambda)$ is always equal to λ, it is clear that the eigenvalue for which $X_1(x)$ is an eigenfunction must be the least eigenvalue. We denote it by λ_1.

We now show that the function which minimizes the functional $G(X)$ in the class of admissible functions satisfying all conditions listed so far and the additional condition

$$\int_0^l \rho X_1 X\,dx=0$$

is an eigenfunction corresponding to the eigenvalue which is second in magnitude (starting with λ_1).

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† The proof of the existence of a solution of this problem as well as of other variational problems discussed in this chapter can be found in the supplement by I. M. Gelfand and G. A. Sukhomlinov to the book by M. A. Lavrentiev and L. A. Liusternik, *Principles of the Calculus of Variations*, vol. I, part II (1935) (in Russian). For an English source see Courant-Hilbert, *Methods of Mathematical Physics*, vol. I (1953), Interscience, New York–London. It is shown in courses on the calculus of variations that if we enlarge the class of admissible functions by requiring that its elements be functions which are once continuously differentiable only, our problem would again have a solution which would necessarily be twice continuously differentiable and would therefore coincide with the solution of this problem in the class of twice continuously differentiable functions.

In fact, the minimizing function must satisfy the Lagrange-Euler equation for the functional

$$G(X)-\lambda H(X)-\mu\int_0^l \rho X_1 X\,dx.$$

This equation can, in the case considered, be written in the form

$$(pX')'-qX-\lambda\rho X+\mu\rho X_1=0, \qquad (6,22)$$

where λ and μ are certain constants.

We now show that $\mu=0$. For proof we write down equation (1, 22) with λ and X replaced by λ_1 and X_1, respectively,

$$(pX_1')'-qX_1+\lambda_1\rho X_1=0. \qquad (7,22)$$

We now multiply (6, 22) by X_1, (7, 22) by X, subtract one from the other and integrate the result from zero to l. Repeating the integration by parts carried out in the proof of orthogonality and making use of the fact that $\int_0^l \rho X_1 X_2 dx=0$, we get

$$\mu\int_0^l \rho X_1^2 dx=0,$$

whence

$$\mu=0.$$

Consequently, equation (6, 22) is equivalent to the equation

$$(pX')'-qX+\lambda\rho X=0,$$

and X is an eigenfunction. We denote this eigenfunction by X_2. We now show that of all eigenvalues different from λ_1 the eigenvalue λ_2 corresponding to X_2 is closest to λ_1. Clearly, $\lambda_2\geqslant\lambda_1$, for an increase in the number of conditions imposed on the admissible functions can only result in an increase of the minimum of $G(X)$. λ_2 cannot be equal to λ_1; otherwise, $X_2(x)$ would be equal to $\pm X_1(x)$ (cf. Theorem 1) which contradicts the condition $\int_0^l X_1 X\,dx=0$. Consequently, $\lambda_2>\lambda_1$.

We now show that there are no eigenvalues between λ_2 and λ_1. In fact, if there existed three eigenvalues $\lambda_2>\tilde{\lambda}>\lambda_1$ corresponding to the eigenfunctions $X_2, \tilde{X}, X_1$, then, as is easily seen, $\tilde{X}$ rather than X_2 would be the solution of the variational problem just considered (cf. Theorem 3).

It can be shown in an entirely analogous manner that the function $X_n(x)$ which minimizes $G(x)$ in the class of twice continuously differentiable functions satisfying equations (5, 22) and the conditions

$$H(X)=1, \quad \int_0^l \rho X_j X\,dx=0 \quad (j=1,2,\dots,n-1)$$

(where $X_j(x)$ is the jth eigenfunction) is the eigenfunction corresponding to the eigenvalue which is nth in magnitude starting with λ_1.

We thus have a method for finding successive eigenvalues and eigenfunctions. Since we show in the sequel that

$$\lambda_n \to \infty \quad \text{as} \quad n \to \infty,$$

it follows that our method enables us to find all eigenvalues and eigenfunctions.

4. There exists a method for finding the nth eigenvalue and the nth eigenfunction without finding all the preceding eigenvalues and eigenfunctions. This method is given by the following *theorem of Courant*:

Let $\phi_1(x), \phi_2(x), \dots, \phi_{n-1}(x)$ be an arbitrary system of continuous functions on the interval $[0, l]$. We denote by $\lambda(\phi_1, \dots, \phi_{n-1})$ the minimum of the functional $G(X)$ in the class of twice continuously differentiable functions which vanish at zero and l and satisfy the following additional conditions:

$$H(X)=1, \tag{8, 22}$$

$$\int_0^l \rho\phi_i X\,dx=0 \quad (i=1,2,\dots,n-1). \tag{9, 22}$$

Then the nth eigenvalue λ_n of the eigenvalue problem considered above is equal to the least upper bound of the values of $\lambda(\phi_1, \phi_2, \dots, \phi_{n-1})$ for all possible choices of the functions $\phi_1, \dots, \phi_{n-1}$.

Before proving this theorem we note that if X_i denotes the ith eigenfunction and $k \neq j$, then

$$G(X_k, X_j)=0.$$

In fact,

$$G(X_k, X_j)=\int_0^l [pX_k'X_j'+qX_kX_j]\,dx$$

$$=pX_k'X_j\Big|_0^l-\int_0^l X_j[(pX_k')'-qX_k]\,dx=\lambda_k\int_0^l \rho X_jX_k\,dx=0.$$

Proof of Courant's theorem. Since, by our previous results,

$$\lambda(X_1, \dots, X_{n-1})=\lambda_n,$$

it suffices to show that for an arbitrary choice of $\phi_1, \dots, \phi_{n-1}$

$$\lambda(\phi_1, \dots, \phi_{n-1}) \leqslant \lambda_n.$$

We show that for an arbitrary system of functions $\phi_1, \dots, \phi_{n-1}$ it is possible to find an admissible function $\tilde{X}(x)$ satisfying the conditions (5, 22), (8, 22) and (9, 22) for which

$$G(\tilde{X}) \leqslant \lambda_n.$$

This will imply that $\lambda(\phi_1, \dots, \phi_{n-1}) \leqslant \lambda_n$,

and our theorem will be proved.

Any function $\tilde{X}(x)$ of the form

$$\tilde{X}(x) = \sum_{k=1}^{n} c_k X_k(x)$$

vanishes at zero and l and is twice continuously differentiable regardless of the choice of the c_k. We now choose the c_k so that $\tilde{X}(x)$ satisfies conditions (8, 22) and (9, 22). Putting $\tilde{X}$ in (8, 22) and making use of the fact that $H(X_i, X_k) = 0$ for $i \neq k$ (orthogonality property of eigenfunctions), we get

$$H(\tilde{X}) = \int_0^l \rho \tilde{X}^2 dx = \sum_{k=1}^{n} c_k = 1. \qquad (10, 22)$$

The conditions (9, 22) yield the system of equations

$$\int_0^l \rho \phi_i \tilde{X} dx = \sum_{k=1}^{n} c_k \int_0^l \rho \phi_i X_k dx = 0 \quad (i = 1, 2, \dots, n-1).$$

This is a system of $n-1$ linear homogeneous equations in n unknowns c_k. This system always has non-trivial solutions. We now make use of (10, 22) to normalize one such solution. Using the normalized c_k, we write down $\tilde{X}(x)$. We now compute $G(\tilde{X})$:

$$\begin{aligned} G(\tilde{X}) &= \int_0^l [p(\Sigma c_k X_k')^2 + q(\Sigma c_k X_k)^2]\, dx \\ &= \int_0^l \left(p \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l X_k' X_l' + q \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l X_k X_l\right) dx \\ &= \sum_{k=1}^{n} c_k^2 G(X_k) + \sum_{k \neq l} c_k c_l G(X_k, X_l) = \sum_{k=1}^{n} c_k^2 G(X_k) \\ &= \sum_{k=1}^{n} c_k^2 \lambda_k \leqslant \lambda_n \sum_{k=1}^{n} c_k^2 = \lambda_n,\dagger \end{aligned}$$

which is what we set out to prove.

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† For $\lambda_1 < \lambda_2 < \dots < \lambda_n$.

Remark. Instead of seeking the minimum of the functional $G(X)$ under the conditions (8, 22) and (9, 22) we could seek the minimum of the quotient $G(X)/H(X)$ under the conditions (9, 22). The minimal value is the same in both cases. In the second case the minimizing function is determined only up to a constant multiplier, however.

5. We now study the effect on the eigenvalues of a change of the coefficients of equation (1, 22) and of the region in which solutions are sought.

(*a*) *A shift of the coefficients $p(x)$ and $q(x)$ in a definite direction results in a shift of the eigenvalues in the same direction.* More precisely, if

$$(pX')' - qX + \lambda\rho X = 0,$$

$$(p^*X')' - q^*X + \lambda\rho X = 0,$$

and

$$p(x) \leqslant p^*(x), \quad q(x) \leqslant q^*(x),$$

then

$$\lambda_n \leqslant \lambda_n^*,$$

where λ_n and λ_n^* are the nth eigenvalues of the first and second of the above equations, respectively.

The proof follows directly from the fact that

$$G(X) = \int_0^l (pX'^2 + qX^2)\,dx \leqslant \int_0^l (p^*X'^2 + q^*X^2)\,dx = G^*(X).$$

Therefore

$$\lambda(\phi_1, \ldots, \phi_{n-1}) \leqslant \lambda^*(\phi_1, \ldots, \phi_{n-1}),$$

since the class of allowable functions $X(x)$ has not changed. Consequently, $\lambda_n \leqslant \lambda_n^*$.

(*b*) *A shift of the coefficient $\rho(x)$ in a definite direction results in a shift of the eigenvalues in the opposite direction.*

Let $\rho(x) \leqslant \rho^*(x)$ and let the remaining coefficients of the equation remain unchanged. Then, for every function $X(x)$,

$$G(X) = G^*(X) \quad \text{and} \quad H(X) \leqslant H^*(X).$$

Hence

$$G(X)/H(X) \geqslant G^*(X)/H^*(X). \tag{11, 22}$$

Every function $X(x)$ satisfying the conditions (9, 22) for certain $\phi_i(x)$ will satisfy analogous conditions

$$\int_0^l \rho^*(x)\,\phi^*(x)\,X(x)\,dx = 0$$

if we put

$$\phi^*(x) = \frac{\rho(x)}{\rho^*(x)}\,\phi(x).$$

This result and the inequality (11, 22) imply that

$$\lambda(\phi_1, \ldots, \phi_{n-1}) \geqslant \lambda^*(\phi_1^*, \ldots, \phi_{n-1}^*).$$

And since $$\rho^*(x) \geqslant \rho(x) > \rho_0 > 0,$$

the totality of all $(\phi_1(x), \ldots, \phi_{n-1}(x))$ coincides with the totality of all $(\phi_1^*(x), \ldots, \phi_{n-1}^*(x))$. Consequently, $\lambda_n \geqslant \lambda_n^*$.

In equation (17, 1) of a vibrating string, $\rho(x)$ denotes the density of the string and $p = T$ the tension.

(*c*) *The decrease in length of the interval* $[0, l]$ *does not result in smaller eigenvalues.* More precisely, if in the eigenvalue problem under consideration the interval $[0, l]$ is replaced by an interval $[0, l^*]$ with $l^* < l$, and if the eigenvalues of the new problem are denoted by λ_n^*, then $\lambda_n^* \geqslant \lambda_n$.

In fact, $\lambda^*(\phi_1, \ldots, \phi_{n-1})$, which in the new problem takes the part of $\lambda(\phi_1, \ldots, \phi_{n-1})$, will coincide with the minimum of the functional $G(X)$ defined for the interval $[0, l]$ if the class of allowable functions is required to satisfy not only conditions (8, 22) and (9, 22) but also the conditions $X(x) \equiv 0$ for $l^* < x < l$. But imposition of additional conditions decreases the class of admissible functions and this can only result in an increase in the minimum of the functional.†

Consequently, $$\lambda^*(\phi_1, \ldots, \phi_{n-1}) \geqslant \lambda(\phi_1, \ldots, \phi_{n-1})$$

so that $$\lambda_n^* \geqslant \lambda_n.$$

Reverting to the example considered in § 20, we can now derive the following well-known connexion between the length of a string and the pitch of its basic tone: the shorter the string the higher the frequency of its eigenvibrations (which is equal to $k\pi/l$) and, consequently, the pitch of the sound it produces.

6. The method used in investigating the eigenvalues of the eigenvalue problem for equation (1, 22) with the boundary conditions

$$X(0) = 0, \quad X(l) = 0 \tag{5, 22}$$

can be applied in investigating the eigenvalues of the eigenvalue problem for the equation (1, 22) with the boundary conditions

$$X'(0) = 0, \quad X'(l) = 0, \tag{12, 22}$$

or $$X'(0) - \sigma_0 X(0) = 0, \quad X'(l) + \sigma_l X(l) = 0, \tag{13, 22}$$

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† If we require that the functions $X(x)$ which are twice continuoulsy differentiable on the interval $[0, l]$ vanish for $l^* \leqslant x \leqslant l$, then we require at the same time that, in addition to $X(x)$, its first two derivatives vanish at $x = l^*$. However, it can be shown that this additional requirement does not change the minimum of $G(X)$ on the interval $[0, l^*]$.

or with one type (of the types indicated above) of boundary conditions at one end-point of the interval $(0, l)$ and another type at the other end-point.

The following analogue of the theorem in para. 4 enables us to investigate eigenvalues for the case of boundary conditions of the type (13, 22).

Let $\phi_1(x), \phi_2(x), \ldots, \phi_{n-1}(x)$ *be an arbitrary system of continuous functions on the interval* $[0, l]$. *We denote by* $\lambda(\phi_1, \ldots, \phi_{n-1})$ *the minimum of the functional*

$$\int_0^l (pX'^2 + qX^2)\,dx + \sigma_0 p(0)\,X^2(0) + \sigma_l p(l)\,X^2(l) \qquad (14, 22)$$

in the class of twice continuously differentiable functions satisfying the conditions

$$H(X) = 1, \quad \int_0^l \rho\phi_i X\,dx = 0 \quad (i = 1, 2, \ldots, n-1). \qquad (15, 22)$$

Then the nth eigenvalue λ_n *of the eigenvalue problem under consideration is equal to the least upper bound of the values of* $\lambda(\phi_1, \ldots, \phi_{n-1})$ *for all possible choices of continuous functions* $\phi_1, \ldots, \phi_{n-1}$.

Using this theorem it is possible to investigate the dependence of the eigenvalues on $p(x)$, $q(x)$, $\rho(x)$, σ_0, σ_l and l, as was done in the case of the fastened string.

If we were to take for the functions $\phi_1, \ldots, \phi_{n-1}$ the first $n-1$ eigenfunctions $X_1, \ldots, X_{n-1}$ of the problem under consideration, then the function minimizing the functional (14, 22) under the conditions (15, 22) would be the nth eigenfunction of this problem and the minimum of the functional would be equal to the nth eigenvalue.

If $\sigma_0 = 0$ and $\sigma_l = 0$, we are led to the eigenvalue problem for equation (1, 22) under the boundary conditions (12, 22). In this case, the nth eigenfunction will minimize $G(X)$ in the class of twice continuously differentiable functions satisfying the conditions

$$H(X) = 1, \quad \int_0^l \rho X_j X\,dx = 0 \quad (j = 1, 2, \ldots, n-1)$$

($X_1, \ldots, X_{n-1}$ are the first $n-1$ eigenfunctions of this problem), which are the same as in the case of fastened end-points. However, we do not now require the admissible functions to satisfy any conditions at the end-points of the interval $(0, l)$. The function which solves this variational problem satisfies automatically the conditions (12, 22). This is the so-called 'free problem'. It is the problem of a vibrating string which is free, i.e. not fastened at its end-points. We wish to observe that to say that the string is not fastened at its end-points means only that each end-point can move along a line perpendicular to the position of equilibrium of the string and not that these two end-points can come closer to each other.

7. If the admissible functions are not required to be continuous at some interior point c of the interval $(0, l)$, then the class of admissible functions is increased. This can only have the effect of decreasing $\lambda(\phi_1, \phi_2, \dots, \phi_{n-1})$ and, consequently, λ_n. The corresponding tone produced by the string becomes lower. This corresponds to the string snapping at some interior point c. Then the end-points of both parts of the string, while remaining each on one and the same line perpendicular to the position of the fixed, tight string, can move freely along this line. The corresponding eigenfunction X_n will have a discontinuity of the first kind at c and

$$X_n'(c+0) = 0, \quad X_n'(c-0) = 0.$$

It follows from our previous discussion that the tones of the string will become lower in comparison with the corresponding tones of the whole string.

§ 23. Justification of the Fourier method

1. Here we again limit ourselves to the consideration of boundary conditions of the form (5, 22), for in all other cases one can apply entirely analogous considerations.

We give an estimate of λ_n depending on n. We denote the maxima of the functions $p(x)$, $q(x)$ and $\rho(x)$ on $[0, l]$ by $p_{\max}$, $q_{\max}$ and $\rho_{\max}$, respectively. Similarly, we denote the minima of these functions on the same interval by $p_{\min}$, $q_{\min}$ and $\rho_{\min}$. We now consider together with equation (1, 22) of the preceding section the following two equations with constant coefficients:

$$p_{\max} X'' - q_{\max} X + \lambda \rho_{\min} X = 0, \tag{1, 23}$$

$$p_{\min} X'' - q_{\min} X + \lambda \rho_{\max} X = 0. \tag{2, 23}$$

The results of the preceding section imply that

$$\underline{\lambda}_n \leqslant \lambda_n \leqslant \overline{\lambda}_n, \tag{3, 23}$$

where $\overline{\lambda}_n$ denotes the nth eigenvalue of equation (1, 23) and $\underline{\lambda}_n$ the nth eigenvalue of equation (2, 23). However, equations (1, 23) and (2, 23) can be integrated in closed form and the values $\overline{\lambda}_n$ and

$\underline{\lambda}_n$ can be computed exactly. Solving equation (1, 23), say, and finding the particular solution of this equation satisfying the initial conditions $X(0)=X(l)=0$, we get

$$\frac{\overline{\lambda}_n \rho_{\min} - q_{\max}}{p_{\max}} = \frac{n^2\pi^2}{l^2}.$$

Whence

$$\overline{\lambda}_n = C_1 n^2 + C_2, \quad \text{where } C_1 \text{ and } C_2 \text{ are independent of } n.$$

Similarly, $$\underline{\lambda}_n = c_1 n^2 + c_2.$$

Substituting these values in (3, 23), we obtain

$$c_1 n^2 + c_2 \leqslant \lambda_n \leqslant C_1 n^2 + C_2. \tag{4, 23}$$

This implies, among others, the existence of infinitely many eigenvalues (and, consequently, of infinitely many eigenfunctions) of equation (1, 22) and their unbounded increase for $n \to \infty$.

2. We now investigate the behavior of the eigenfunctions for increasing n. To this end we simplify equation (1, 22) by means of the substitution

$$t = \int_0^x \phi(x)\,dx, \quad u = \frac{1}{\psi(x)} X. \tag{5, 23}$$

We choose the functions $\phi(x)$ and $\psi(x)$ so that, as a result of the application of the substitution (5, 23), equation (1, 22) takes the form

$$u''(t) + \lambda u = R(t)\,u. \tag{6, 23}$$

Application of the substitution (5, 23) with arbitrary $\phi(x)$ and $\psi(x)$ takes equation (1, 22) into the equation

$$\frac{d^2u}{dt^2} + \frac{(\phi\psi p)' + \phi\psi' p}{\phi^2\psi p}\frac{du}{dt} + \lambda\rho\frac{1}{\phi^2 p}u = \frac{\psi q - (\psi' p)'}{\phi^2\psi p}u.$$

We now choose $\phi(x)$ and $\psi(x)$ so that this equation has the form (6, 23). For this it is necessary to determine the functions $\phi(x)$ and $\psi(x)$ from the system of equations

$$\rho/\phi^2 p = 1, \quad (\phi\psi p)' + \phi\psi' p = 0.$$

Solving this system, we get

$$\phi = (\rho/p)^{\frac{1}{4}}; \quad \psi = c/(\rho p)^{\frac{1}{4}}.$$

It is now possible to obtain equation (6, 23) (from equation (1, 22)) by means of the substitution

$$t = \int_0^x (\rho/p)^{\frac{1}{4}}\,dx, \quad u = (\rho p)^{\frac{1}{4}} X, \tag{7, 23}$$

say. Here $R(t)$ is a bounded function, for $\phi^2\psi p \neq 0$.

We are to find a solution of equation (6, 23) on the interval $0 < t < l_1$, where $l_1 = \int_0^l (\rho/p)^{\frac{1}{2}}\, dx$. It is easily seen that the boundary conditions for $u(t)$ are the same as those for $X(x)$, namely,

$$u(0) = 0, \quad u(l_1) = 0.$$

If $X_n(x)$ denotes the eigenfunction of equation (1, 22) corresponding to the eigenvalue λ_n, then u_n is the eigenfunction of equation (6, 23) corresponding to the same eigenvalue.

If

$$\int_0^l \rho X_n^2\, dx = 1,$$

then, as is easily seen,

$$\int_0^{l_1} u_n^2(t)\, dt = 1. \tag{8, 23}$$

We now give asymptotic formulas for $u_n(t)$ for $t \to \infty$. To this end we consider the non-homogeneous equation in the function $z(t)$:

$$z'' + \lambda z = R(t)\, u \quad (\lambda > 0), \tag{*}$$

where $u(t)$ denotes a solution of equation (6, 23) for λ appearing in (*). The general solution of our equation is

$$z(t) = C_1 \cos \lambda^{\frac{1}{2}} t + C_2 \sin \lambda^{\frac{1}{2}} t + \lambda^{-\frac{1}{2}} \int_0^t R(\tau)\, u(\tau) \sin \lambda^{\frac{1}{2}} (t - \tau)\, d\tau.†$$

If we put $C_1 = u(0)$ and $C_2 = u'(0)/\lambda^{\frac{1}{2}}$, then, for $t = 0$, $z(t)$ will satisfy the same initial conditions as $u(t)$. Consequently, in view of the theorem on the uniqueness of the solution of the Cauchy problem for equation (*), $z(t)$ will be identically equal to $u(t)$ and we obtain for $u(t)$ the following integral equation:

$$u(t) = u(0) \cos \lambda^{\frac{1}{2}} t + u'(0)\, \lambda^{-\frac{1}{2}} \sin \lambda^{\frac{1}{2}} t + \lambda^{-\frac{1}{2}} \int_0^t R(\tau)\, u(\tau) \sin \lambda^{\frac{1}{2}} (t - \tau)\, d\tau.$$

Now let λ coincide with the nth eigenvalue and let $v_n(t)$ be a solution of equation (6, 23) for $\lambda = \lambda_n$ satisfying the initial conditions

$$v_n(0) = 0, \quad v_n'(0) = \lambda_n^{\frac{1}{2}}.$$

Such function $v_n(t)$ will satisfy the integral equation

$$v_n(t) = \sin \lambda_n^{\frac{1}{2}} t + \lambda^{-\frac{1}{2}} \int_0^t R(\tau)\, v_n(\tau) \sin \lambda_n (t - \tau)\, d\tau. \tag{9, 23}$$

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† R. Courant, *Differential and Integral Calculus*, vol. II, Interscience, New York. Also, cf. the author's *Lectures on the Theory of Ordinary Differential Equations*, Gostekhizdat (1949), p. 141 (in Russian).

Apart from sign, this function differs from the normalized eigenfunction $u_n(t)$ only by the factor

$$\left[\int_0^{l_1} v_n^2(t)\,dt\right]^{-\frac{1}{2}} = \frac{1}{N_n}.$$

In what follows we show that

$$N_n \to (2/l)^{\frac{1}{2}} \quad \text{as} \quad n \to \infty.$$

We prove first that all functions $v_n(t)$ are bounded by some constant which is independent of n. To this end we denote the maximum of $|v_n(t)|$ on $0 \leqslant t \leqslant l_1$ by M_n. We then have by equation (9, 23)

$$|v_n(t)| \leqslant 1 + \lambda_n^{-\frac{1}{2}} M_n \int_0^{l_1} |R(\tau)|\,d\tau$$

and, consequently,

$$M_n \leqslant 1 + M_n \lambda_n^{-\frac{1}{2}} \int_0^{l_1} |R(\tau)|\,d\tau.$$

Hence
$$M_n\left(1 - \lambda_n^{-\frac{1}{2}} \int_0^{l_1} |R(\tau)|\,d\tau\right) \leqslant 1$$

and
$$M_n \leqslant \frac{1}{1 - \lambda_n^{-\frac{1}{2}} \int_0^{l_1} |R(\tau)|\,d\tau} = 1 + O\,(\lambda_n^{-\frac{1}{2}}). \tag{10, 23}$$

Since $\lambda_n \to \infty$ with n (cf. para. 1), this inequality proves the boundedness of the functions $v_n(t)$.

We shall need similar estimates for the first and second derivatives of the eigenfunctions. To obtain such estimates we differentiate the integral equation (9, 23). We then obtain

$$v_n'(t) = \lambda_n^{\frac{1}{2}} \cos \lambda_n^{\frac{1}{2}} t + \int_0^{l_1} R(\tau)\, v_n(\tau) \cos \lambda_n^{\frac{1}{2}} (t - \tau)\, d\tau,$$

$$v_n''(t) = -\lambda_n \sin \lambda_n^{\frac{1}{2}} t - \lambda_n^{\frac{1}{2}} \int_0^{t} R(\tau)\, v_n(\tau) \sin \lambda_n^{\frac{1}{2}}(t - \tau)\, d\tau + R(t)\, v_n(t),$$

whence
$$|v_n'(t)| \leqslant \lambda_n^{\frac{1}{2}} + O(1), \quad |v_n''(t)| \leqslant \lambda_n + O(\lambda_n^{\frac{1}{2}}). \tag{11, 23}$$

We now compute $\int_0^{l_1} v_n^2(t)\,dt$, i.e. we find the multiplicative factor by which the functions $v_n(t)$ (and, consequently, their derivatives) differ from the normalized eigenfunctions $u_n(t)$ (and their corresponding derivatives). From (9, 23) we have

$$v_n^2(t) = \sin^2 \lambda_n^{\frac{1}{2}} t + O(\lambda_n^{-\frac{1}{2}}).$$

Hence
$$\int_0^{l_1} v_n^2(t)\,dt = \tfrac{1}{2} l_1 - \sin 2\lambda_n^{\frac{1}{2}} l_1 + O(\lambda_n^{-\frac{1}{2}}) = \tfrac{1}{2} l_1 + O(\lambda_n^{-\frac{1}{2}}).$$

This yields immediately estimates for $u_n(t)$ and its derivatives analogous to (10, 23) and (11, 23):

$$\left.\begin{aligned} u_n(t) &= (2/l_1)^{\frac{1}{2}} + O(\lambda_n^{-\frac{1}{2}}), \\ u_n'(t) &= (2/l_1)^{\frac{1}{2}} \lambda_n^{\frac{1}{2}} + O(1), \\ u_n''(t) &= (2/l_1)^{\frac{1}{2}} \lambda_n + O(\lambda_n^{\frac{1}{2}}). \end{aligned}\right\} \qquad (12, 23)$$

Making use of the substitution (7, 23), we now obtain from the above results corresponding results for $X_n(x)$, namely, the boundedness of the eigenfunctions and the same order of growth of the derivatives for $n \to \infty$ as in the case of the $u_n(t)$.

3. We now go over to the investigation of the problem of expanding an arbitrary continuous function defined on the interva $0 \leqslant x \leqslant l$ in a power series

$$\sum_{n=1}^{\infty} c_n X_n(x), \qquad (13, 23)$$

where $X_1(x), \ldots, X_n(x), \ldots$ are the eigenfunctions of equation (1, 22).

By means of the technique used for ordinary trigonometric series it is easy to show that if the series (13, 23) converges uniformly to the function $f(x)$, then the coefficients c_n are equal to the Fourier coefficients of the function $f(x)$ with respect to the system of functions $X_1, \ldots, X_n, \ldots$, i.e.

$$c_n = \int_0^l \rho(x) f(x)\, X_n(x)\, dx \qquad (14, 23)$$

(cf. the end of § 21).

We now associate with every integrable function $f(x)$ its 'Fourier series' of the form (13, 23), where the coefficients c_n are defined by (14, 23), and we investigate the convergence of this series.

We first show that *for every function $f(x)$ which is integrable together with its square over the interval $[0, l]$ and piecewise continuous*† *in this interval, the series* (13, 23) *converges to $f(x)$ 'in the mean', i.e. that*

$$\int_0^l \rho(x) \left[f(x) - \sum_{n=1}^{N} c_n X_n(x) \right]^2 dx \xrightarrow[N \to \infty]{} 0.$$

A system of functions which are orthogonal with respect to some weight function $\rho(x)$ and have the indicated property is *complete.*

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† I.e. having a finite number of points of discontinuity.

To prove this theorem we first assume that $f(x)$ *is continuously differentiable and that it satisfies the conditions* $f(0)=f(l)=0$. We introduce the notations

$$f_N(x)=f(x)-\sum_{n=1}^{N}c_nX_n(x),\quad \delta_N^2=\int_0^l\rho(x)f_N^2(x)\,dx,\quad \phi_N(x)=\frac{f_N(x)}{\delta_N}.$$

We must show that $\delta_N^2\to 0$ for $N\to\infty$.

Since
$$\int_0^l\rho\phi_N^2(x)\,dx=1,$$
and, in addition,
$$\int_0^l\rho\phi_N(x)X_n(x)\,dx=0\quad(n=1,\ldots,N),$$
$\phi_N(x)$ is seen to be one of the admissible functions of the variational problem considered in para. 3 of the preceding section. For that problem, the value of the minimum of $G(X)$ is equal to λ_{N+1}. Consequently
$$G[\phi_N(x)]\geqslant\lambda_{N+1}.$$

We now compute $G(\phi_N)$. Using the notations of the preceding section we find

$$G(\phi_N)=\int_0^l(p\phi_N'^2+q\phi_N^2)\,dx=\frac{1}{\delta_N^2}\int_0^l(pf_N'^2+qf_N^2)\,dx$$
$$=\frac{1}{\delta_N^2}\int_0^l\left[p\left(f'-\sum_{n=1}^{N}c_nX_n'\right)^2+q\left(f-\sum_{n=1}^{N}c_nX_n\right)^2\right]dx$$
$$=\frac{1}{\delta_N^2}\left[G(f)-2\sum_{n=1}^{N}c_nG(f,X_n)+\sum_{n=1}^{N}\sum_{m=1}^{N}c_nc_mG(X_n,X_m)\right]\geqslant\lambda_{N+1}.\tag{15, 23}$$†

In paras. 3 and 4 of § 22 we established the following results:

$$G(X_n,X_n)=G(X_n)=\lambda_n,\quad G(X_n,X_m)=0\quad\text{for}\quad n\neq m;$$
$$G(f,X_n)=\int_0^l(pf'X_n'+qfX_n)\,dx=pX_n'f\Big|_0^l-\int_0^l[(pX_n')'-qX_n]f\,dx$$
$$=\int_0^l\lambda_n\rho X_nf\,dx=\lambda_nc_n.$$

Here we have made use of the restrictions imposed on the values of $f(x)$ at the end-points.

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† Cf. footnote on p. 128.

Substituting the values of the functionals in (15, 23) we find that

$$\delta_N^{-2}\left[G(f)-\sum_{n=1}^{N}\lambda_n c_n^2\right]\geqslant\lambda_{N+1},$$

whence

$$\delta_N^2\leqslant\frac{G(f)-\sum\limits_{n=1}^{N}\lambda_n c_n^2}{\lambda_{N+1}}. \tag{16, 23}$$

According to (4, 23) there is only a finite number of negative λ_n. Consequently, the numerator of the right side of (16, 23) is bounded for all N. Since

$$\lambda_{N+1}\to\infty \quad \text{as} \quad N\to\infty,$$

it follows that

$$\delta_N^2\to 0 \quad \text{as} \quad N\to\infty,$$

i.e. the series (13, 23) converges in the mean for every differentiable function which vanishes at $x=0$ and $x=l$.

We can easily free ourselves from the restrictions imposed on $f(x)$ if we note that, given a function which is integrable together with its square and piecewise continuous, it is possible to approximate it to an arbitrary degree of closeness by means of a function satisfying our restrictions (these restrictions appear in italics above), for the integral of the square of the difference of the two functions can be made arbitrarily small. Thus, let $f(x)$ be a function defined on the interval $[0, l]$ and piecewise continuous and integrable, together with its square, over this interval. Further, let $f^*(x)$ be a twice continuously differentiable function vanishing at zero and l such that

$$\int_0^l [f(x)-f^*(x)]^2\,dx<\epsilon_1.$$

Finally, let N be so large that

$$\int_0^l \rho(x)\left[f^*(x)-\sum_{n=1}^{N}c_n^* Xn(x)\right]^2 dx<\epsilon_2,$$

where c_n^* are the Fourier coefficients of $f^*(x)$. Then

$$\begin{aligned}
&\int_0^l \rho(x)\left[f(x)-\sum_{n=1}^{N}c_n^* X_n(x)\right]^2 dx\\
&\qquad\leqslant\int_0^l \rho(x)\left[|f(x)-f^*(x)|+\left|f^*(x)-\sum_{n=1}^{N}c_n^* X_n(x)\right|\right]^2 dx\\
&\qquad\leqslant\epsilon_1+\epsilon_2+2\int_0^l \rho(x)\,|f(x)-f^*(x)|\,.\left|f^*(x)-\sum_{n=1}^{N}c_n^* X_n(x)\right| dx\\
&\qquad\leqslant\epsilon_1+\epsilon_2+2(\epsilon_1\epsilon_2)^{\frac{1}{2}}.
\end{aligned}$$

In estimating the last integral we made use of Schwarz's inequality.

We have thus shown that for every function $f(x)$ which is integrable together with its square there exist numbers N and c_n such that

$$\int_0^l \rho(x)\left[f(x)-\sum_{n=1}^{N} c_n X_n(x)\right]^2 dx \tag{17, 23}$$

is arbitrarily small. We know, however, that an integral of the form (17, 23) takes on its least value when the c_n are the Fourier coefficients of the function $f(x)$. Consequently, if the c_n in (17, 23) are taken as these very coefficients, the value of the integral can only decrease.

Using the orthogonality property of the functions $X_n(x)$, it is easily shown that, if the c_n are the Fourier coefficients of $f(x)$, then

$$\delta_N^2=\int_0^l \rho[f(x)]^2\,dx-\sum_{n=1}^{N} c_n^2 \geqslant 0 \quad \text{(Bessel's inequality).}$$

Consequently, the completeness condition for a system of functions can be written in the form of the following equality

$$\sum_{n=1}^{\infty} c_n^2=\int_0^l \rho[f(x)]^2\,dx \quad \text{(Parseval's equality).}$$

Parseval's equality implies, among others, that the Fourier coefficients of every square integrable function go to zero with increasing n.

We now prove the following result. *If*

(*a*) *$f(x)$ is three times continuously differentiable and has a piecewise continuous fourth derivative*

(*b*) *$f(x), f'(x), f''(x)$ satisfy the conditions*

$$f(0)=f(l)=0, \quad f'(0)=f'(l)=0,$$

$$f''(0)=f''(l)=0, \quad f'''(0)=f'''(l)=0;$$

(*c*) *the coefficients of the original equation are four times continuously differentiable, then*

(A) *the Fourier coefficients of $f(x)$ for $n\to\infty$ are of the order of smallness of $1/n^4$*

(B) *the Fourier series* (13, 23) *converges uniformly and is twice termwise differentiable.*

To simplify our proof we again carry out the substitution (7, 23) and replace the function $f(x)$ by $f_1(t)$. In view of the restrictions imposed on the coefficients of the equation, the function $f_1(t)$ will satisfy the same smoothness conditions as the function $f(x)$, and it will vanish together with its first three derivatives at the end-points of the interval $[0, l]$, and the function $R(t)$ in equation (6, 23) will be twice continuously differentiable.

We shall be interested in the order of smallness of

$$c_n = \int_0^l \rho f(x) X_n(x)\,dx = \int_0^{l_1} f_1(t)\,u_n(t)\,dt \tag{18, 23}$$

for $n \to \infty$. The function $u_n(t)$ satisfies the equation

$$u_n(t) = N_n \sin \lambda_n^{\frac{1}{2}} t + \lambda_n^{-\frac{1}{2}} \int_0^t R(\tau)\,u_n(\tau) \sin \lambda_n^{\frac{1}{2}}(t-\tau)\,d\tau, \tag{19, 23}$$

where $N_n = \frac{1}{2} l_1 + O[1/(\lambda_n)^{\frac{1}{2}}]$ (cf. (9, 23) and (12, 23)).

Using this equation we now show that it is possible to find functions

$$u_n^*(t), \quad u_n^{**}(t), \quad u_n^{***}(t), \quad u_n^{****}(t)$$

$$(u_n^{*\prime}(t) = u_n(t), \quad u_n^{**\prime}(t) = u_n^*(t), \quad u_n^{***\prime}(t) = u_n^{**}(t), \quad u_n^{****\prime}(t) = u_n^{***}(t)),$$

which, for $n \to \infty$, differ from the functions

$$-N_n \lambda_n^{-\frac{1}{2}} \cos \lambda_n^{\frac{1}{2}} t, \quad -N_n \lambda_n^{\frac{1}{2}} \sin \lambda_n^{\frac{1}{2}} t, \quad N_n \lambda_n^{-\frac{3}{2}} \cos \lambda_n^{\frac{1}{2}} t, \quad N_n \lambda_n^{2} \sin \lambda_n^{\frac{1}{2}} t$$

(obtained by repeated integration of $N_n \sin \lambda_n^{\frac{1}{2}} t$) by infinitesimals of order higher than (in the case of the last function—at least as high as) the numerical factors of the trigonometric functions. Then, integrating (as in the case of ordinary trigonometric series) formula (18, 23) by parts and making use of the boundary conditions for $f_1(t)$, we can show that the Fourier coefficients c_n are of the order of smallness indicated above.

If we put

$$u_n^*(t) = -N_n \lambda_n^{-\frac{1}{2}} \cos \lambda_n^{\frac{1}{2}} t - \lambda_n^{-\frac{1}{2}} \int_0^t R(\tau)\,u_n(\tau) \cos \lambda_n^{\frac{1}{2}}(t-\tau)\,d\tau$$
$$+ \lambda_n^{-1} \int_0^t R(\tau)\,u_n(\tau)\,d\tau - N_n \lambda_n^{-\frac{3}{2}} R(0), \tag{20, 23}$$

it is clear that

$$u_n^{*\prime}(t) = u_n(t) \quad \text{and} \quad u_n^*(t) = -N_n \lambda_n^{-\frac{1}{2}} \cos \lambda_n^{\frac{1}{2}} t + O(\lambda_n^{-1}).$$

We now tranform the second integral in (20, 23) by substituting in it for $u_n(\tau)$ the expression (19, 23) and by integrating by parts the first of the terms obtained:

$$\int_0^t R(\tau)\,u_n(\tau)\,d\tau = N_n \int_0^t R(\tau) \sin \lambda_n^{\frac{1}{2}} \tau\,d\tau + \lambda_n^{-\frac{1}{2}} \int_0^t R(\tau)\,d\tau \int_0^\tau R(s)\,u_n(s) \sin \lambda_n^{\frac{1}{2}}(\tau - s)\,ds$$
$$= -N_n \lambda_n^{-\frac{1}{2}} R(t) \cos \lambda_n^{\frac{1}{2}} t + N_n \lambda_n^{-\frac{1}{2}} R(0) + N_n \lambda_n^{-\frac{1}{2}} \int_0^t R'(\tau) \cos \lambda_n^{\frac{1}{2}}\,d\tau$$
$$+ \lambda_n^{-\frac{1}{2}} \int_0^t R(\tau)\,d\tau \int_0^\tau R(s)\,u_n(s) \sin \lambda_n^{\frac{1}{2}}(\tau - s)\,ds. \tag{21, 23}$$

Putting (21, 23) in (20, 23) we obtain an expression for $u_n^*(t)$ which is convenient for the purpose of further integration:

$$u_n^*(t) = -N_n \lambda_n^{-\frac{1}{2}} \cos \lambda_n^{\frac{1}{2}} t - \lambda_n^{-1} \int_0^t R(\tau)\,u_n(\tau) \cos \lambda_n^{\frac{1}{2}}(t-\tau)\,d\tau$$
$$+ \lambda_n^{-\frac{3}{2}} \left\{ -N_n R(t) \cos \lambda_n^{\frac{1}{2}} t + N_n \int_0^t R_t'(\tau) \cos \lambda_n^{\frac{1}{2}} \tau\,d\tau \right.$$
$$\left. + \int_0^t R(\tau)\,d\tau \int_0^\tau R(s)\,u_n(s) \sin \lambda_n^{\frac{1}{2}}(\tau - s)\,ds \right\}.$$

We now put

$$\begin{aligned} u_n^{**}(t) = -N_n\lambda_n^{-1}\sin\lambda_n^{\frac12}t - \lambda_n^{-\frac32}\int_0^t R(\tau)u_n(\tau)\sin\lambda_n^{\frac12}(t-\tau)\,d\tau \\ -N_n\lambda_n^{-\frac32}\int_0^t R(\tau)\cos\lambda_n^{\frac12}\tau\,d\tau + N_n\lambda_n^{-\frac32}\int_0^t d\tau\int_0^\tau R'(s)\cos\lambda_n^{\frac12}s\,ds \\ +\lambda_n^{-\frac32}\int_0^t d\tau\int_0^\tau R(s)\,ds\int_0^s R(\sigma)u_n(\sigma)\sin\lambda_n^{\frac12}(s-\sigma)\,d\sigma. \qquad (22,\ 23)\end{aligned}$$

It is easy to check that, first, $u_n^{**\prime}(t) = u_n^*(t)$,

and, secondly, $u_n^{**}(t) = -N_n\lambda_n^{-1}\sin\lambda_n^{\frac12}t + O(\lambda^{-\frac32})$.

Further, carrying out as before one integration by parts in each of the three last terms in (22, 23), we find easily that their order of smallness for $n\to\infty$ is not less than $1/\lambda_n^2$ (here an essential point is the boundedness of $R''(t)$ on the interval $[0, l]$). Consequently,

$$u_n^{**}(t) = -N_n\lambda_n^{-1}\sin\lambda_n^{\frac12}t - \lambda_n^{-\frac32}\int_0^t R(\tau)u_n(\tau)\sin\lambda_n^{\frac12}(t-\tau)\,d\tau + \lambda_n^{-2}\Phi_n(t),$$

where the $\Phi_n(t)$ are bounded by a common constant for all n.

Putting

$$\begin{aligned} u_n^{***}(t) = N\lambda_n^{-\frac32}\cos\lambda_n^{\frac12}t \\ +\lambda_n^{-2}\int_0^t R(\tau)u_n(\tau)\left[\cos\lambda_n^{\frac12}(t-\tau)-1\right]d\tau + \lambda_n^{-2}\int_0^t \Phi_n(\tau)\,d\tau,\end{aligned}$$

we see that

$$u_n^{***\prime}(t) = u_n^{**}(t) \quad \text{and} \quad u_n^{***}(t) = N\lambda_n^{-\frac32}\cos\lambda_n^{\frac12}t + O(\lambda_n^{-2}).$$

Integrating once more from zero to t we find the function

$$u_n^{****}(t) = N\lambda_n^{-2}\sin\lambda_n^{\frac12}\,t + O(\lambda_n^{-2}),$$

which is of the desired order of smallness.

4. We now prove the following fundamental theorem: *The 'Fourier series' (13, 23) of a function $f(x)$ which is continuously differentiable on the interval $[0, l]$ converges to this function (on this interval) absolutely and uniformly.*

It suffices to show that this series converges absolutely and uniformly. In fact, since this series converges 'in the mean' to $f(x)$, it cannot converge uniformly to any other function.

Making use of Cauchy's inequality we can write

$$\begin{aligned}\sum_{k=n}^{n+s}|c_kX_k| = \sum_{k=n}^{n+s}|c_k\lambda_k^{\frac12}|\,|X_k\lambda_k^{-\frac12}| \\ \leqslant \left[\sum_{k=n}^{n+s}c_k^2\lambda_k\right]^{\frac12}\left[\sum_{k=n}^{n+s}\frac{X_k^2}{\lambda_k}\right]^{\frac12} \leqslant \left[\sum_{k=1}^{n+s}c_k^2\lambda_k\right]^{\frac12}\left[\sum_{k=n}^{n+s}\frac{X_k^2}{\lambda_k}\right]^{\frac12}.\end{aligned}$$

We now apply the inequality (16, 23) to get an estimate for the first factor, and in the second factor we take out from under the summation sign

$$\max|X_k(x)| = M.$$

Then
$$\sum_{k=n}^{n+s} |c_k X_k| \leqslant [G(f) - \lambda_{n+s+1}\delta_{n+s}^2]^{\frac{1}{2}} M\left(\sum_{k=n}^{n+s} \lambda_k^{-1}\right)^{\frac{1}{2}}$$
$$\leqslant [G(f)]^{\frac{1}{2}} M\left(\sum_{k=n}^{n+s} \lambda_k^{-1}\right)^{\frac{1}{2}}.$$

Since, according to para. 1 of this section,

$$\lambda_k \geqslant c_1 k^2 + c_2,$$

it follows that $\lambda^{-k} \leqslant (c_1 k^2 + c_2)^{-1}$ and the series $\sum_{k=1}^{\infty} \lambda_k^{-1}$ converges. Consequently, for any $\epsilon > 0$,

$$\sum_{k=n}^{n+s} \frac{1}{\lambda_k} < \frac{\epsilon^2}{M^2 G(f)}$$

for n sufficiently large and for an arbitrary s. Hence, for the same n,

$$\sum_{k=n}^{n+s} |c_k X_k| < \epsilon,$$

i.e. the series $\Sigma c_k X_k(x)$ converges absolutely and uniformly.

5. We now return to the problem formulated in § 21. We must justify the possibility of finding a solution of the equation

$$A(t)\frac{\partial^2 u}{\partial t^2} + C(x)\frac{\partial^2 u}{\partial x^2} + D(t)\frac{\partial u}{\partial t} + E(x)\frac{\partial u}{\partial x} + [F_1(t) + F_2(x)]\,u = 0$$

with sufficiently smooth coefficients under boundary conditions

$$u(t, 0) = 0, \quad u(t, l) = 0$$

and initial conditions

$$u(0, x) = \phi_0(x), \quad u_t'(0, x) = \phi_1(x)$$

in the form of a series

$$u(t, x) = \sum_{k=1}^{\infty} X_k(x)\,[A_k T_k^*(t) + B_k T_k^{**}(t)], \tag{23,23}$$

where
$$\sum_{k=1}^{\infty} A_k X_k(x) = \phi_0(x), \tag{24, 23}$$

$$\sum_{k=1}^{\infty} B_k X_k(x) = \phi_1(x). \tag{25, 23}$$

As in § 21, we denote by $T_k^*(t)$ and $T_k^{**}(t)$ solutions of the equation

$$A(t)\,T''(t) + D(t)\,T'(t) + F_1(t)\,T + \lambda T = 0, \tag{26, 23}$$

satisfying the initial conditions

$$T_k^*(0)=1, \quad T_k^{*\prime}(0)=0,$$
$$T_k^{**}(0)=0, \quad T_k^{**\prime}(0)=1.$$

We assume (cf. § 21) that the original equation is hyperbolic and that $A(t)>a_0>0$. It follows that we can apply to equation (26, 23) all transformations which enabled us to reduce the equation for the function $X_k(x)$ in §§ 23 and 24 to the form

$$u''+\lambda u=R(\tau)\,u.$$

The boundedness of the totality of functions $T_k^*(t)$ and $T_k^{**}(t)$ can be shown in the same way in which it has been shown for the functions $X_k(x)$ in para. 2 of this section. The same holds for the order of growth of the derivatives of these functions for $k\to\infty$.

The functions $T_k^*(t)$ and $T_k^{**}(t)$ can be obtained by means of a transformation independent of k from the functions $z_k^*(\tau)$ and $z_k^{**}(\tau)$ satisfying the differential equation

$$z_k+\lambda_k z=Q(\tau)z \tag{27, 23}$$

and the conditions $$z_k^*(0)=a^*, \quad z_k^{*\prime}(0)=b^*,$$

and $$z_k^{**}(0)=a^{**}, \quad z_k^{**\prime}(0)=b^{**},$$

respectively. Here the constants a^*, b^*, a^{**} and b^{**} depend on the coefficients of equation (26, 23) only and can be easily computed. In each case, equation (27, 23) can be reduced to an integral equation in the manner shown in para. 2 of this section. From the integral equation we can obtain directly estimates for the maximum of the absolute value of the solution as well as of its derivatives

If the coefficients $A(t)$, $D(t)$ and $F_1(t)$ are constant (as, for example, in the case of the equation of the homogeneous vibrating string), the corresponding estimates can be easily obtained by finding explicit solutions of the proper equations satisfying the required initial conditions.

Since the functions $T_k^*(t)$ and $T_k^{**}(t)$ are bounded, the series (23, 23) will converge uniformly if the series (24, 23) and (25, 23) converge uniformly, i.e. if the functions $\phi_0(x)$ and $\phi_1(x)$ are continuously differentiable and satisfy the conditions

$$\phi_0(0)=\phi_0(l)=0, \quad \phi_1(0)=\phi_1(l)=0.$$

For the series (23, 23) to satisfy the initial conditions and the original equation it is necessary that it be twice termwise differentiable with respect to both variables. From what has already been shown it follows that the first and second derivatives of the functions $X_k(x)$ and $T_k(t)$ do not exceed a quantity of the order of $c\lambda_n$. If the functions $\phi_0(x)$ and $\phi_1(x)$ satisfy the supplementary conditions of smoothness formulated in para. 3 of this section (small print), then, for some c_1, their Fourier coefficients A_k and B_k do not exceed the quantity c_1/λ_n^2. Therefore, for a suitable M, the series $M\sum_{k=1}^{\infty} 1/\lambda_n$ majorizes the series obtained by differentiating the series (23, 23) termwise two times. This proves that these series converge uniformly.

Thus the series (23, 23) is actually the solution of equation (1, 21) and satisfies the conditions set down.

Problem 1. Show that the solution of the boundary-value problem for equation (1, 21) depends continuously on the functions $\phi_0(x)$ and $\phi_1(x)$ (in the same sense in which this was shown in § 19 for the equation of a vibrating string).

Problem 2. Using the result of the preceding problem show the existence of a generalized solution of this equation if the functions $\phi_0(x)$ and $\phi_1(x)$ are continuous together with their first derivatives and vanish at the end-points of the interval.

§ 24. Another justification of the Fourier method

It is possible to show the completeness of the system of eigenfunctions of the eigenvalue problem and the fundamental properties of this system without solving variational problems. This can be done by reducing our boundary-value problem to a Fredholm integral equation of the second kind. The reduction is realized by means of the *Green function* which we are about to construct.

1. We consider the problem of finding a solution of the equation

$$(pX')' - qX = f(x) \tag{1, 24}$$

in the interval $(0, l)$ satisfying the conditions

$$X(0) = X(l) = 0. \tag{2, 24}$$

Together with equation (1, 24) we consider the equation

$$(pY')' - qY = g_\epsilon(x, x_0) \tag{3, 24}$$

whose left side is the same as the left side of equation (1, 24), and whose free term is defined as follows:

$$g_\epsilon(x, x_0) = \begin{cases} 1/\epsilon & \text{for } x_0 - \frac{1}{2}\epsilon < x < x_0 + \frac{1}{2}\epsilon, \\ 0 & \text{for all remaining values of } x. \end{cases}$$

Here ϵ and x_0 are certain parameters; $\epsilon > 0$, $0 \leqslant x_0 \leqslant l$. We assume that we know the solution $Y_\epsilon(x, x_0)$ of this equation satisfying the initial conditions (2, 24) and depending on the parameters ϵ and x_0.

We multiply equation (1, 24) by Y_ϵ and equation (3, 24), with Y replaced by Y_ϵ, by X. We subtract the second equation from the first and integrate the result from zero to l. We get

$$\int_0^l [(pX')'Y_\epsilon - (pY_\epsilon')'X]\,dx = \int_0^l [Y_\epsilon(x, x_0)f(x) - X(x)\,g_\epsilon(x, x_0)]\,dx.$$

Since the functions $X(x)$ and $Y_\epsilon(x, x_0)$ vanish at the ends of the interval of integration, the left side of this equality vanishes. This is easy to see if we integrate twice by parts:

$$\int_0^l (pX')'\,Y_\epsilon dx = pX'Y_\epsilon\Big|_0^l - \int_0^l pX'\,Y_\epsilon'dx$$

$$= -pY_\epsilon'X\Big|_0^l + \int_0^l (pY_\epsilon')'\,Xdx = \int_0^l (pY_\epsilon')'\,Xdx.$$

Consequently

$$\int_0^l Y_\epsilon(x, x_0)f(x)\,dx = \int_0^l g_\epsilon(x, x_0)\,X(x)\,dx$$

$$= \frac{1}{\epsilon}\int_{x_0-\frac{1}{2}\epsilon}^{x_0+\frac{1}{2}\epsilon} X(x)\,dx \approx X(x_0). \qquad (4, 24)$$

If we assume that $Y_\epsilon(x, x_0)$ goes to some limit function $G(x, x_0)$ for $\epsilon \to 0$, then, effecting formal transition to the limit for $\epsilon \to 0$ on both sides of the equality (4, 24), we get

$$X(x_0) = \int_0^l G(x, x_0)f(x)\,dx. \qquad (5, 24)$$

We call the limit function $G(x, x_0)$ the *Green function* for equation (1, 24).

These non-rigorous considerations are useless when it comes to proving facts. We therefore give a definition of the Green function which does not depend on the above heuristic considerations and we show its existence as well as the correctness of the formula (5, 24).

Before giving an exact definition of the Green function we explain what properties the limit of $Y_\epsilon(x, x_0)$ must have, should it exist. After replacing Y by $Y_\epsilon(x, x_0)$, we integrate the identity (3, 24) with respect to x from $x_0 - \delta$ to $x_0 + \delta$, where $\delta > \frac{1}{2}\epsilon$. We obtain

$$\int_{x_0-\delta}^{x_0+\delta} \{[pY_\epsilon'(x, x_0)]' - qY_\epsilon(x, x_0)\}\,dx = \int_{x_0-\delta}^{x_0+\delta} g_\epsilon(x, x_0)\,dx = 1.$$

The first term can be integrated explicitly and the preceding equality takes the form

$$pY'_\epsilon(x,x_0)\Big|_{x_0-\delta}^{x_0+\delta}-\int_{x_0-\delta}^{x_0+\delta}qY_\epsilon(x,x_0)\,dx=1.$$

Assuming that the formal transition to the limit for $\epsilon\to 0$ and for a fixed δ is legitimate, we obtain the equality

$$p(x_0+\delta)\,G'_x(x_0+\delta,x_0)-p(x_0-\delta)\,G'_x(x_0-\delta,x_0)$$
$$-\int_{x_0-\delta}^{x_0+\delta}q(x)\,G(x,x_0)\,dx=1,$$

which holds for any $\delta>0$. Going over to the limit for $\delta\to 0$ and assuming that $p(x)$, $q(x)$ and $G(x,x_0)$ are continuous functions, we obtain the equality

$$p(x_0)\,[G'_x(x_0+0,x_0)-G'_x(x_0-0,x_0)]=1,$$

from which it is clear that the x-derivative $G'_x(x,x_0)$ of the Green function 'jumps' at $x=x_0$ by $1/p(x)$.

2. We now give a formal definition of the Green function for equation (1, 24) and we prove the existence of this function.

By the Green function for equation (1, 24) *with initial conditions* (2, 24) *we mean a function* $G(x,s)$, *defined on the square* $0\leqslant x\leqslant l$, $0\leqslant s\leqslant l$ *and satisfying the following conditions:*

1°. *For* $x\neq s$, $G(x,s)$, *regarded as a function of* x, *is twice continuously differentiable and satisfies the homogeneous equation*

$$[pG'(x,s)]'_x-qG(x,s)=0. \tag{6, 24}$$

2°. $$G(0,s)=G(l,s)=0.$$

3°. $G(x,s)$ *is continuous on the square* $0\leqslant x\leqslant l$, $0\leqslant s\leqslant l$, *and* $G'_x(x,s)$, *regarded as a function of* x, *jumps for* $x=s$ *by* $1/p(s)$, i.e.

$$G'_x(s+0,s)-G'_x(s-0,s)=1/p(s).$$

When proving the existence of such a function we assume that $q(x)>0$, since $\lambda=0$ is not an eigenvalue of the equation

$$(pX')'-qX+\lambda\rho X=0$$

with the initial conditions (2, 24).

With this assumption, we prove the existence of the Green function simply by constructing it. Thus, let $X_1(x)$ be a non-trivial solution of equation (6, 24) satisfying the condition

$$X_1(0)=0,$$

and $X_2(x)$ a solution of the same equation satisfying the condition

$$X_2(l)=0.$$

In view of the assumptions made, the solutions $X_1(x)$ and $X_2(x)$ are linearly independent. Otherwise they would be proportional to each other and each of them would vanish for $x=0$ and $x=l$. However, this is impossible, since $\lambda=0$ is not an eigenvalue. We put

$$G(x,s)=\begin{cases} A(s)\,X_1(x) & (0\leqslant x<s), \\ B(s)\,X_2(x) & (s<x\leqslant l). \end{cases} \tag{7,24}$$

Then conditions 1° and 2° are satisfied for arbitrary $A(s)$ and $B(s)$.

We now choose $A(s)$ and $B(s)$ so that condition 3° holds. The condition of continuity of $G(x,s)$ for $x=s$ implies

$$A(s)\,X_1(s)=B(s)\,X_2(s),$$

whence

$$A(s)=c(s)\,X_2(s),$$
$$B(s)=c(s)\,X_1(s).$$

We require the jump of the derivative for $x=s$ to have the value $1/p(s)$:

$$G'_x(s-0,s)=c(s)\,X_2(s)\,X'_1(s),$$
$$G'_x(s+0,s)=c(s)\,X_1(x)\,X'_2(s),$$

whence

$$c(s)=\frac{1}{p(s)}\,\frac{1}{X_1(s)\,X'_2(s)-X'_1(s)\,X_2(s)}.$$

The denominator $p(s)\,[X_1(s)\,X'_2(s)-X_2(s)\,X'_1(s)]$ does not depend on s. In fact, the expression in square brackets is the Wronskian $\Delta(X_1,X_2)$ of two linearly independent solutions $X_1(s)$ and $X_2(s)$. By a well-known formula we have

$$\Delta(X_1,X_2)=\Delta_0\exp\left\{-\int\frac{p'(x)}{p(x)}\,dx\right\}=\frac{\Delta_0}{p(x)},$$

which implies that $c(s)$ is constant.

Hence, the Green function has the form

$$\left.\begin{aligned} G(x,s)&=\frac{1}{\Delta_0}\,X_2(s)\,X_1(x) \quad \text{for} \quad 0\leqslant x\leqslant s, \\ G(x,s)&=\frac{1}{\Delta_0}\,X_1(s)\,X_2(x) \quad \text{for} \quad s\leqslant x\leqslant l, \end{aligned}\right\} \tag{8,24}$$

and its existence has been proved.

It is clear from (8, 24) *that the Green function is symmetric in its arguments*:

$$G(x,s) = G(s,x).$$

We now prove formula (5, 24) for any twice continuously differentiable function $X(x)$ satisfying an equation of the form (1, 24) and vanishing at $x=0$ and $x=l$. For proof we multiply (1, 24) by $G(x,s)$ and (6, 24) by $X(x)$. We subtract the second equation from the first and integrate the resulting difference from zero to s and from s to l, respectively. We then obtain two equalities:

$$\int_0^s \{[pX'(x)]' G(x,s) - [pG'_x(x,s)]'_x X(x)\}\,dx = \int_0^s G(x,s) f(x)\,dx, \tag{9, 24}$$

$$\int_s^l \{[pX'(x)]' G(x,s) - [pG'_x(x,s)]'_x X(x)\}\,dx = \int_s^l G(x,s) f(x)\,dx. \tag{10, 24}$$

Integrating the first term on the left side of (9, 24) twice by parts, we get

$$\begin{aligned}
&\int_0^s [pX'(x)]'G(x,s)\,dx \\
&\quad = pX'(x)\,G(x,s)\Big|_0^s - \int_0^s pX'(x)\,G'_x(x,s)\,dx \\
&\quad = p(s)\,X'(s)\,G(s,s) - p(x)\,G'_x(x,s)\,X(x)\Big|_0^s \\
&\qquad\qquad + \int_0^s [p(x)\,G'_x(x,s)]'_x\,X(x)\,dx \\
&\quad = p(s)\,X'(s)\,G(s,s) - p(s)\,X(s)\,G'_x(s-0,s) + \int_0^s [pG'_x(x,s)]'_x\,X(x)\,dx.
\end{aligned}$$

Substituting this in the left side of (9, 24), we obtain

$$p(s)\,X'(s)\,G(s,s) - p(s)\,X(s)\,G'_x(s-0,s) = \int_0^s G(x,s) f(x)\,dx. \tag{11, 24}$$

Treated similarly, (10, 24) yields

$$-p(s)\,X'(s)\,G(s,s) + p(s)\,X(s)\,G'_x(s+0,s) = \int_0^l G(x,s) f(x)\,dx. \tag{12, 24}$$

Adding equations (11, 24) and (12, 24) and taking into account property 3° of the Green function, we get

$$X(s) = \int_0^l G(x,s) f(x)\,dx. \tag{5, 24}$$

Reversing the order of the considerations which led us to (5, 24), we can show that, for any continuous function $f(x)$, the function $X(x)$ defined by (5, 24) satisfies equation (1, 24). From the form of the right side of equation (5, 24) it follows that the function $X(x)$ defined by (5, 24) vanishes for $x = 0$ and $x = l$.

3. We now show how one can use the Green function for equation (1, 24) to reduce the eigenvalue problem considered in the preceding sections to an integral equation. To this end we write the fundamental equation (1, 24) in the form

$$(pX')' - qX = -\lambda\rho X, \tag{13,24}$$

and, putting $f(x) = -\lambda\rho X$, we apply it to formula (5, 24). We obtain the equality

$$X(s) + \lambda\int_0^l G(x,s)\,\rho(x)\,X(x)\,dx = 0, \tag{14,24}$$

which represents a homogeneous Fredholm equation of the second kind with symmetrizable kernel and with parameter λ.

The kernel of (14, 24) can be symmetrized by multiplying equation (14, 24) by $[\rho(s)]^{\frac{1}{2}}$. Then this equation becomes an equation in the unknown function $[\rho(s)]^{\frac{1}{2}}X(s)$ with a symmetric kernel $G(x,s)\,[\rho(x)\,\rho(s)]^{\frac{1}{2}}$. In view of (5, 24), equation (13, 24), together with the boundary conditions $X(0) = X(l) = 0$ and equation (14, 24), are equivalent in the sense that every solution of (13, 24) which vanishes at $X = 0$ and $x = l$ is an eigenfunction of (14, 24) and conversely.

On the other hand, the theorems on the existence of eigenvalues and eigenfunctions, on the orthogonality of the system of eigenfunctions, and the expansion theorem, all of which were proved in §§ 22 and 23, hold for equations of the form (14, 24) (cf. Courant-Hilbert, *Methods of Mathematical Physics*, vol. I). This implies the existence, orthogonality and expansion theorems proved in §§ 22 and 23. It is true that to prove the possibility of expansion of $f(x)$ one must require the continuity of its second derivative to make sure that it can be represented in the form (5, 24).

It is possible to define the Green function which reduces the solution of a differential equation to the solution of an integral equation for other types of boundary conditions as well as for equations in many variables. However, as a rule, effective expressions for the Green function can be obtained only for equations and boundary conditions of very special form.

§ 25. Investigation of the vibration of a membrane

1. In § 1 we considered as an example the equation of a vibrating membrane:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}. \tag{1, 25}$$

When in a position of equilibrium, let the membrane coincide with some region G of the (x, y) plane having a piecewise smooth boundary Γ. Then the function $u(t, x, y)$ which determines the vibrations of the membrane must satisfy equation (1, 25) and the initial conditions

$$\left.\begin{aligned} u(0,x,y)&=\phi_0(x,y) \quad \text{(initial displacement)},\\ u_t'(0,x,y)&=\phi_1(x,y) \quad \text{(initial velocity)}, \end{aligned}\right\} \tag{2, 25}$$

when the point (x, y) is in G. At points of the boundary Γ of the region G the function $u(t, x, y)$ must satisfy certain boundary conditions of the type considered in § 1.

We consider the simplest case, that of a membrane rigidly fastened along the edge, i.e. the boundary condition

$$u(t,x,y)=0 \quad \text{for} \quad (x,y) \text{ in } G. \tag{3, 25}$$

Solving the problem by the method of separation of variables, we put

$$u(t,x,y)=T(t)\,v(x,y).$$

In analogy to the one-dimensional case we get the following equations for the functions $T(t)$ and $v(x, y)$:

$$T''+\lambda T=0, \tag{4, 25}$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\lambda v=0. \tag{5, 25}$$

The eigenfunctions of the latter equation in the region G with the boundary condition $v(x, y)\,|_\Gamma=0$ represent a complete orthogonal system

$$v_1(x,y), v_2(x,y), \ldots, v_n(x,y), \ldots,$$

corresponding to a non-decreasing sequence of eigenvalues

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots.$$

In distinction to the case of a single variable, equality signs may actually turn up in this sequence, i.e. so-called 'multiple' eigenvalues can occur. The eigenfunctions corresponding to such eigenvalues may also be supposed orthogonal (cf. Courant-Hilbert, *Methods of Mathematical Physics*, vol. I).

Expanding the functions $\phi_0(x,y)$ and $\phi_1(x,y)$ in power series in the functions $v_n(x,y)$

$$\left.\begin{aligned}\phi_0(x,y)&=\sum_{n=1}^{\infty}A_n v_n(x,y),\\ \phi_1(x,y)&=\sum_{n=1}^{\infty}B_n v_n(x,y)\end{aligned}\right\} \tag{6,25}$$

and choosing, as at the end of § 23, linearly independent solutions of equation (4, 25) satisfying the conditions

$$T^*(0)=1, \quad T^{*\prime}(0)=0,$$
$$T^{**}(0)=0, \quad T^{**\prime}(0)=1,$$

we obtain the series

$$u(t,x,y)=\sum_{n=1}^{\infty}v_n(x,y)\,[A_n T_n^*(t)+B_n T_n^{**}(t)], \tag{7,25}$$

which represents the solution of our problem.

The problem of finding the eigenfunctions of equation (5, 25) for an arbitrary region G is very complicated.

We now take up two special cases in which the eigenfunctions of equation (5, 25) can be found by the method of separation of variables. An analogous procedure can be adopted for a greater number of independent variables. These cases can be fully investigated by reducing them to the one-dimensional eigenvalue problem by means of the following lemma:

LEMMA. *Let* $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$ *be a complete system of functions orthogonal and normalized with respect to the weight function* $\rho_1(x)$ *on the interval* $[a,b]$. *Further, let there be given for every* n $(n=1,2,\ldots)$ *a complete system of functions orthogonal and normalized with respect to the weight function* $\rho_2(y)$ *on the interval* $[c,d]$. *Let the latter functions be*

$$\psi_{1n}(y), \psi_{2n}(y), \ldots, \psi_{mn}(y), \ldots. \tag{8,25}$$

It is assumed that the functions $\rho_1(x)$ *and* $\rho_2(y)$ *are bounded and positive. Then the functions*

$$X_{nm}(x,y)=\phi_n(x)\,\psi_{nm}(y)$$

form a complete system of orthogonal and normalized functions with respect to the weight function $\rho(x,y)=\rho_1(x)\rho_2(y)$ *in the rectangle* $a<x<b$, $c<y<d$, *i.e. the following equalities hold:*

$$\int_c^d\int_a^b \rho(x,y)\,X_{nm}(x,y)\,X_{n'm'}(x,y)\,dx\,dy=\begin{cases}1 & \text{for } n=n', m=m',\\ 0 & \text{for } n\neq n', \text{ or } m\neq m'.\end{cases} \tag{9,25}$$

Also, if we put

$$c_{nm}=\int_c^d\int_a^b \rho(x,y)f(x,y)\,X_{nm}(x,y)\,dx\,dy,$$

Parseval's equality

$$\int_c^d\int_a^b \rho(x,y)\,[f(x,y)]^2\,dx\,dy=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{nm}^2 \tag{10, 25}$$

holds for every continuous square integrable function $f(x,y)$.

Proof. Formulas (9, 25) are obviously true. To prove (10, 25) we put

$$\int_a^b \rho_1(x)f(x,y)\,\phi_n(x)\,dx=g_n(y).$$

It is then clear that

$$\int_c^d \rho_2(y)\,g_n(y)\,\psi_{nm}(y)\,dy=c_{nm}$$

and that
$$\int_c^d \rho_2(y)\,g_n^2(y)\,dy=\sum_{m=1}^{\infty}c_{nm}^2,$$

for the systems $\psi_{nm}(y)$ are complete for every n, and $g_n(y)$ is square integrable. The latter statement can be verified by applying Schwarz's inequality. But then

$$\int_c^d\int_a^b \rho(x,y)\,[f(x,y)]^2\,dx\,dy=\int_c^d \rho_2(y)\sum_{n=1}^{\infty}g_n^2(y)\,dy$$
$$=\sum_{n=1}^{\infty}\int_c^d \rho_2(y)\,g_n^2(y)\,dy=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}^2,$$

which is what we set out to prove.

2. We now consider the first special case, that of the *vibrations of a rectangular membrane.*

Let the region G be the rectangle $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$. Separating the variables in equation (5, 25), we put

$$v(x,y)=X(x)\,Y(y).$$

On substitution of this function, equation (5, 25) takes the form

$$X''Y+XY''+\lambda XY=0.$$

Dividing by XY and transferring X''/X to the right side of the equation we obtain the equation

$$Y''/Y+\lambda=-X''/X$$

which is equivalent to the following two ordinary differential equations:

$$X''+\alpha X=0,$$

$$Y''+\beta Y=0.$$

Here α is an arbitrary constant and $\beta=\lambda-\alpha$. According to the boundary conditions (3, 25), the first of the above two ordinary differential equations is to be solved for

$$X(0)=X(a)=0,$$

and the second, for $\quad Y(0)=Y(b)=0.$

To satisfy these conditions we must assume that $\alpha>0$ and $\beta>0$ (cf. § 20). Repeating the considerations of § 20, we find that the eigenvalues and eigenfunctions of both equations are, respectively,

$$\alpha_n=\frac{n^2\pi^2}{a^2};\quad X_n(x)=(2/a)^{\frac{1}{2}}\sin\frac{n\pi}{a}x;$$

$$\beta_n=\frac{m^2\pi^2}{b^2};\quad Y_m(x)=(2/b)^{\frac{1}{2}}\sin\frac{m\pi}{b}y.$$

According to our lemma, the system of functions

$$v_{nm}(x,y)=2/(ab)^{\frac{1}{2}}\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y \tag{11, 25}$$

is a complete system of orthogonal and normalized solutions of equation (5, 25) over the rectangle $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$ for the boundary conditions (3, 25) (here $Y_{nm}(y)=Y_m(y)$ for arbitrary n). To every function $v_{nm}(x,y)$ there corresponds the eigenvalue

$$\lambda_{mn}=\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right).$$

It is clear that, if the numbers a and b are commensurable, we can obtain the same λ for different choices of n and m, i.e., for different eigenfunctions. This is therefore an example in which multiple eigenvalues can occur.

The problem of expansion of initial data in a series in the functions (11, 25) is the thoroughly explored problem of expansion in a double Fourier sine series. If, as a result of an odd extension with respect to x and y to the rectangle $|x|\leqslant a$, $|y|\leqslant b$ and a periodic extension to the whole plane, the initial data represent four times continuously differentiable functions, then the coefficients of the expansion (6, 25) go to zero fast enough for the series

(7, 25) to be twice differentiable. Thus, in this case, the use of the Fourier method for the solution of the given problem is fully justified. We see that, just as in the case of the vibrations of a string, arbitrary vibrations of a membrane can be represented by means of superposition of simple, so-called 'eigenvibrations' corresponding to the eigenvalues λ_{nm}.

Of interest are the 'knot curves' of such vibrations. These are the curves along which an eigenfunction corresponding to a given eigenvalue vanishes. We investigate these curves for a rectangular membrane. If the eigenvalue under consideration is not a multiple eigenvalue, i.e. if there corresponds to it a single eigenfunction

$$[2/(ab)^{\frac{1}{2}}] \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y,$$

then the knot curves are simply straight-line segments parallel to the sides of the rectangle. In the case of a multiple eigenvalue, different knot curves correspond to different combinations of the eigenfunctions associated with that eigenvalue, and the knot curves can assume a large variety of forms. In fig. 11 are represented knot curves of a square membrane of unit side length for $\lambda = 5\pi^2$, $10\pi^2$, $13\pi^2$ and $17\pi^2$. Underneath each drawing there appears the corresponding eigenfunction.

3. As our second example we consider the case of a circular membrane. To investigate this example it is essential to write equation (5, 25) in polar coordinates. Putting $x = \rho \cos \phi$, $y = \rho \sin \phi$, we get

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial v}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 v}{\partial \phi^2} + \lambda v = 0. \qquad (12, 25)$$

If we place the center of the circle D which coincides with the position of equilibrium of our membrane at the origin, and if, for the sake of simplicity, we assume the radius of D to be of unit length, then the boundary condition (3, 25) can be written as

$$v(1, \phi) = 0.$$

Using the method of separation of variables, we put

$$v(\rho, \phi) = R(\rho)\, \Phi(\phi).$$

Substituting this expression in (12, 25) and separating variables, we obtain a pair of ordinary differential equations for $R(\rho)$ and $\Phi(\phi)$:

$$\Phi''(\phi) + \alpha \Phi(\phi) = 0, \qquad (13, 25)$$

$$\rho^2 R''(\rho) + \rho R'(\rho) + (\lambda \rho^2 - \alpha) R = 0. \qquad (14, 25)$$

The physical situation involved dictates a periodicity condition for the solutions of equation (13, 25): namely, we are interested only in those solutions of (13, 25) which are periodic with period 2π. Such solutions exist for

$$\alpha = 0, 1^2, 2^2, \ldots, n^2, \ldots.$$

For these values of α

$$\Phi_n(\phi) = A_n \cos n\phi + B_n \sin n\phi.$$

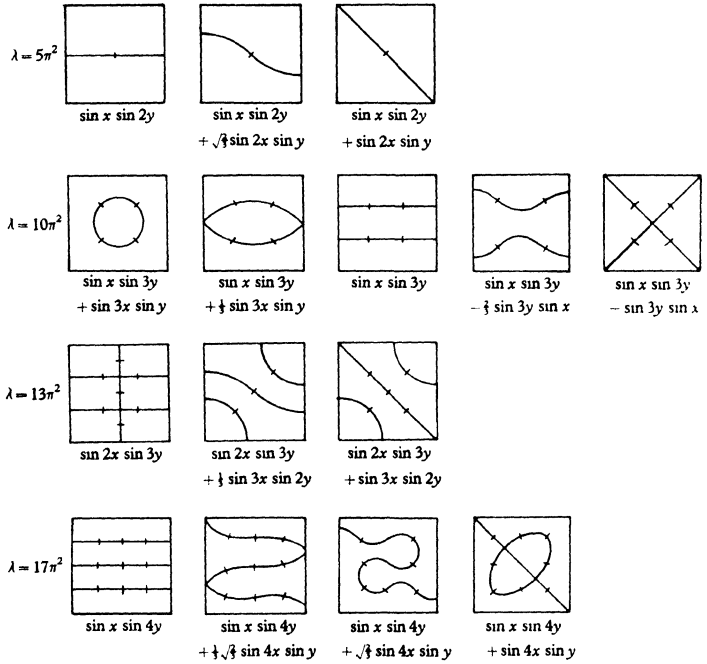
$\lambda = 5\pi^2$
$\sin x \sin 2y$
$\sin x \sin 2y + \sqrt{\tfrac{2}{3}} \sin 2x \sin y$
$\sin x \sin 2y + \sin 2x \sin y$
$\lambda = 10\pi^2$
$\sin x \sin 3y + \sin 3x \sin y$
$\sin x \sin 3y + \tfrac{1}{3} \sin 3x \sin y$
$\sin x \sin 3y$
$\sin x \sin 3y - \tfrac{2}{3} \sin 3y \sin x$
$\sin x \sin 3y - \sin 3y \sin x$
$\lambda = 13\pi^2$
$\sin 2x \sin 3y$
$\sin 2x \sin 3y + \tfrac{1}{3} \sin 3x \sin 2y$
$\sin 2x \sin 3y + \sin 3x \sin 2y$
$\lambda = 17\pi^2$
$\sin x \sin 4y$
$\sin x \sin 4y + \tfrac{1}{3}\sqrt{\tfrac{2}{3}} \sin 4x \sin y$
$\sin x \sin 4y + \sqrt{\tfrac{2}{3}} \sin 4x \sin y$
$\sin x \sin 4y + \sin 4x \sin y$

Figure 11

As our complete system of functions $\Phi_n(\phi)$, orthogonal and normalized on the circumference, we can take the functions

$$\Phi_0 = 1/(2\pi)^{\frac{1}{2}}, \quad \Phi_n^*(\phi) = (2/\pi)^{\frac{1}{2}} \cos n\phi, \quad \Phi_n^{**}(\phi) = (2/\pi)^{\frac{1}{2}} \sin n\phi.$$

We now turn to equation (14, 25). Putting $\alpha = n^2$ and replacing the independent variable ρ by ρ_1,

$$\rho_1 = \rho(\lambda)^{\frac{1}{2}},$$

we obtain Bessel's equation of order n

$$\rho_1^2 R''(\rho_1) + \rho_1 R'(\rho_1) + (\rho_1^2 - n^2) R(\rho_1) = 0.$$

Except for a constant multiplier, the only solution of this equation for $\rho_1 = 0$ (i.e. for $\rho = 0$) is Bessel's function of order n of the first kind: $J_n(\rho_1)$.†

The function $J_n(x)$ (n arbitrary) has infinitely many positive roots $\mu_1^{(n)}, \mu_2^{(n)}, \ldots, \mu_m^{(n)}, \ldots$, so that $J_n(\mu_m^{(n)} x) = 0$.†

For a fixed arbitrary n the system of functions

$$J_n(\mu_m^{(n)} x) \quad (m = 1, 2, \ldots, n, \ldots)$$

is complete and orthogonal with respect to the weight function x on the interval $[0, 1]$:

$$\int_0^1 x J_n(\mu_m^{(n)} x)\, J_n(\mu_{m_1}^{(n)} x)\, dx = 0 \quad \text{for} \quad m \neq m_1.$$

The system of functions

$$\phi_{nm}(x) = J_n(\mu_m^{(n)} x) \Big/ \left[\int_0^1 x[J_n(\mu_m^{(n)} x)]^2\, dx\right]^{\frac{1}{2}}$$

has all the properties of the system of functions

$$J_n(\mu_m^{(n)} x) \quad (m = 1, 2, \ldots, n, \ldots)$$

and is, in addition, normalized.

Without proving these facts† we observe that they are generalizations of the properties of eigenfunctions proved in §§ 22 and 23 in that they involve equations with more complex coefficients than those which appeared in §§ 22 and 23. In fact, equation (14, 25) can be written in the form

$$(\rho R')' - (\alpha/\rho) R + \lambda \rho R = 0,$$

and we see that the first and last coefficients vanish at one end-point of the interval $[0, 1]$, and α/ρ becomes infinite at that end-point. Under the circumstances it can be proved that: to determine a solution to within a constant multiplier it suffices to take as a boundary condition at $\rho = 0$ the condition of boundedness of the solution provided that at $\rho = 1$ we prescribe a condition of one of the types (2, 22).

We require that for $\rho = 1$

$$J_n(\lambda^{\frac{1}{2}} \rho) = 0,$$

i.e. that

$$J_n(\lambda^{\frac{1}{2}}) = 0.$$

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† Cf. G. N. Watson, *A Treatise on the Theory of Bessel Functions* (1944), Macmillan, New York. R. O. Kuzmin, *Bessel Functions*, ONTI (1935) (in Russian).

We observe that, if $\mu_1^{(n)}, \mu_2^{(n)}, \ldots, \mu_m^{(n)}, \ldots$ is the sequence of zeros of the function $J_n(x)$, then the eigenvalues λ of our problem are given by

$$\lambda_{nm} = [\mu_m^{(n)}]^2,$$

and the normalized eigenfunctions of equation (14, 25) are the functions

$$\psi_{nm}(\rho) = J_n(\mu_m^{(n)}\rho) \Big/ \int_0^1 \rho [J_n(\mu_m^{(n)}\rho)]^2 \, d\rho.$$

Applying the lemma of para. 1, we can obtain a complete system of eigenfunctions

$$\psi_{nm}(\rho)\, \Phi_n^*(\phi), \quad \psi_{nm}(\rho)\, \Phi_n^{**}(\phi)$$

of equation (12, 25) and find a solution of our problem by expanding the functions $\phi_0(\rho, \phi)$ and $\phi_1(\rho, \phi)$ in series of the form

$$\phi_0(\rho, \phi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [c_{nm}^* \, \Phi_n^*(\phi) + c_{nm}^{**} \, \Phi_n^{**}(\phi)] \, \psi_{nm}(\rho),$$

$$\phi_1(\rho, \phi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [d_{nm}^* \, \Phi_n^*(\phi) + d_{nm}^{**} \, \Phi_n^{**}(\phi)] \, \psi_{nm}(\rho).$$

Multiplying each term of the first series by a suitable $T^*(t)$ and each term of the second series by a suitable $T^{**}(t)$ and summing the resulting series, we obtain the series (7, 25) which represents the solution of the given problem. If the functions $\phi_0(\rho, \phi)$ and $\phi_1(\rho, \phi)$ are sufficiently smooth, then the sum series converges uniformly and admits of termwise differentiation, provided that the functions ϕ_0 and ϕ_1 satisfy the boundary conditions which are to be satisfied by the solution sought as well as certain additional conditions on the boundary of the circle.

26. Supplementary information concerning eigenfunctions

1. All that has been said so far about the eigenvalue problem for equation (1, 22) carries over in a natural way to the analogous problem for the equations

$$(p_1 u_x')_x' + (p_2 u_y')_y' - qu + \lambda \rho u = 0, \qquad (1, 26)$$

$$(p_1 u_x')_x' + (p_2 u_y')_y' + (p_3 u_z')_z' - qu + \lambda \rho u = 0 \qquad (2, 26)$$

under the assumption that p_i and their derivatives as well as ρ are continuous and that p_i and ρ exceed certain positive constants.

Let D be some region bounded by a piecewise smooth curve L. We seek a non-trivial solution of equation (1, 26) in D satisfying on L the condition

$$u = 0 \qquad (3, 26)$$

or the condition

$$\partial u / \partial n + \sigma u = 0. \qquad (4, 26)$$

Here $\partial/\partial n$ denotes differentiation in the direction of the outer normal and σ is a certain non-negative function defined on the boundary of D. For this problem eigenvalues and eigenfunctions are defined in a manner analogous to that used in our previous work. In case of the boundary condition (3, 26) we construct the functionals

$$H(u)=\iint_D \rho u^2\,dx\,dy,$$

$$G(u)=\iint_D (p_1u_x^2+p_2u_y^2+qu^2)\,dx\,dy,$$

and in the case of the boundary condition (4, 26), the functional

$$G^*(u)=G(u)+\int_L \sigma u^2\,dl=\iint_D (p_1u_x^2+p_2u_y^2+qu^2)\,dx\,dy+\int_L \sigma u^2\,dl.$$

It is then easy to carry over to the present case all theorems on eigenvalues and eigenfunctions proved in § 22. In particular, Courant's theorem on the maximum-minimum property of eigenfunctions holds. The consequences of this theorem concerning the dependence of the eigenvalues on the coefficients of the equation, on the region D, and on the boundary conditions are also valid. As is easily seen, the specific nature of this dependence can be summed up by saying that the nth eigenvalue does not decrease as $\sigma(l)$, p_1, p_2, q and $1/\rho$ increase.

We are led to problems of this type when studying, for instance, vibrations of a membrane. Then properties analogous to those described in paras. 5 (*c*) and 7, § 22, admit of an interesting physical interpretation indicating the nature of the change in frequency of the eigenvibrations of the membrane when the latter is fastened in some parts of the region D (para. 5 (*c*), § 22) or when it is broken (para. 7, § 22). The decrease in frequency resulting from breaks is in agreement with the well-known physical fact that broken objects give off lower tones than unbroken ones.

We mention two other theorems which remain valid for equations (1, 26) and (2, 26). One theorem concerns the completeness of the system of eigenfunctions, and the other refers to the possibility of expanding any function f which satisfies the same boundary conditions on the boundary as the eigenfunctions considered in an absolutely and uniformly converging series in these eigenfunctions. However, if the number of independent variables is greater than one, stricter smoothness conditions must be imposed on the function f for the expansion theorem to hold. For two or three independent variables it suffices to require that the function f have bounded and continuous derivatives of order up to 3. The method of proof of the expansion theorem used in the case of a single independent variable cannot be used for a larger number of independent variables, and methods involving integral equations are used instead.

2. While discussing the behavior of eigenfunctions of equation (1, 22) we have not gone into the question of changes of sign ('zeros') of the function $u_n(x)$ corresponding to the eigenvalue λ_n on the interval $(0, l)$. This matter forms the content of the so-called Sturm oscillation theorems.

It turns out that, first, the nth eigenfunction of equation (1, 22) under the boundary conditions (5, 22) has exactly $n-1$ zeros in the interior of the interval $[0, l]$ and that, secondly, the zeros of the function $u_{n+1}(x)$ and the zeros of $u_n(x)$ alternate, more precisely, that in every interval between two roots of $u_{n+1}(x)$ there lies one root of the function $u_n(x)$ (cf. Courant-Hilbert, *Methods of Mathematical Physics*, vol. I).

As regards the knot curves of the nth eigenfunction of equation (1, 26) under the boundary conditions (3, 26), it can be shown that they divide the fundamental region D into no more than n subregions. It is known that in distinction to the case of a single independent variable the number of these subregions can be smaller than n. In the case of many independent variables no theorems analogous to Sturm's theorem on alternating zeros have been proved for successive eigenfunctions of a regular equation. It is therefore not surprising to learn that the asymptotic behavior of the eigenfunctions over general regions is unknown.

3. Many problems of classical and modern physics lead to the problem of determining the eigenfunctions and eigenvalues of the equation

$$u'' + \lambda u = R(x) u \qquad (5, 26)$$

in either the infinite interval $-\infty < x < \infty$ or in the finite interval $(0, l)$, where the function $R(x)$ becomes infinite at one or at both end-points of the interval.†

Different cases of the theory of expansion in eigenfunctions call for different generalizations. We mention two of the most important cases.

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† We observe that every equation of the form (1, 22) can be put in the form (5, 26) by means of change of variables.

(*a*) *The case of a finite interval:* $0<x<l$, $R(0)=\infty$. In many problems it is required that, instead of a boundary condition at $x=0$, the condition

$$\int_0^l u^2(x,\lambda)\,dx<\infty \tag{6, 26}$$

holds, where $u(x,\lambda)$ is a solution of equation (5, 26). Here it turns out that in some cases not all solutions of the equation satisfy the condition (6, 26). In those cases the condition (6, 26) and the boundary condition at $x=l$ determine uniquely (to within a constant multiplier) the eigenvalues and eigenfunctions; the spectrum turns out to be discrete and Sturm's oscillation theorem holds.

In other cases all solutions of equation (5, 26) turn out to satisfy condition (6, 26), and it is necessary to introduce an additional boundary condition at the point $x=0$. We do not dwell here on the nature of this condition. Under this additional condition and under the condition at $x=l$, the spectrum turns out to be discrete. In both cases the expansion theorem is valid for a large class of functions.

Both possibilities are well illustrated by Bessel's equation

$$y''+\frac{1}{x}y'+\left(s^2-\frac{\nu^2}{x^2}\right)y=0. \tag{7, 26}$$

Its solutions are

$$J_\nu(sx),\, Y_\nu(sx)=\frac{J_\nu(sx)\cos\nu\pi-J_{-\nu}(sx)}{\sin\nu\pi}.$$

As a result of the substitution $y_1=x^{\frac{1}{2}}y$, equation (7, 26) goes over into the equation

$$y_1''+\left(s^2-\frac{\nu^2-\frac{1}{4}}{x^2}\right)y_1=0, \tag*{$(7, 26)_1$}$$

which is of the form (5, 26).

If $\nu>1$, then only the functions $x^{\frac{1}{2}}J_\nu(sx)$ vanish at $x=0$. If $0\leqslant\nu\leqslant1$, all solutions of equation (7, 26) vanish at $x=0$.

In the first case, to obtain eigenfunctions and eigenvalues one need only impose on the function $x^{\frac{1}{2}}J_\nu(sx)$ a single boundary condition at the point l; an example of such condition is the condition

$$\frac{d}{dx}\{x^{\frac{1}{2}}j_\nu(sx)\}_{x=l}-H(x^{\frac{1}{2}}j_\nu(sx))_{x=l}=0. \tag{8, 26}$$

This condition and condition (6, 26) determine the eigenvalues and eigenfunctions.

In the second case (i.e. when $0 \leqslant \nu \leqslant 1$), condition (8, 26) must be supplemented by some condition at the point $x = 0$.

(*b*) *The interval* $(0, \infty)$, $R(x)$ *a continuous function.* In this case physical problems usually lead to the determination of those solutions $u(x)$ of equation (5, 26) which satisfy some boundary condition at $x = 0$ and are bounded for $x \to \infty$. If $R(x)$ is absolutely integrable over the interval $(0, \infty)$, we obtain a so-called *continuous spectrum*, i.e. a continuous sequence of eigenvalues and a family of eigenfunctions $u(x, \lambda)$ which vary continuously with λ. Also, the following generalization of Parseval's equality (i.e. the definition of completeness of a system of eigenfunctions) holds:

Let $f(x)$ *be square integrable over the interval* $(0, \infty)$. *Then*

$$\int_0^\infty f^2(x)\,dx = \int_{-\infty}^\infty F^2(\lambda)\,d\rho(\lambda) \quad \text{(Parseval's equality).}$$

Here $F(\lambda)$ (the generalized Fourier transform of $f(x)$) is the limit in the sense of the mean square of the functions

$$F_n(\lambda) = \int_0^n f(x)\,u(x, \lambda)\,dx,$$

i.e.

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} [F(\lambda) - F_n(\lambda)]^2\,d\rho(\lambda) = 0,$$

$\rho(\lambda)$ being some non-decreasing function.

For the representation

$$f(x) = \int_{-\infty}^{+\infty} u(x, \lambda)\,dg(\lambda)$$

of $f(x)$ in the form of an eigenfunction integral with respect to the parameter λ (analog of the ordinary Fourier integral for the equation $u'' + \lambda u = 0$) to hold, much more restrictive conditions have to be imposed on the function $f(x)$.

For a more detailed discussion of these problems we refer the reader to E. C. Titchmarsh, *Eigenfunction Expansions associated with Second Order Differential Equations*, Oxford, The Clarendon Press, 1946. For a Russian source see B. M. Levitan, *Eigenfunction Expansions*, Gostekhizdat, 1950.

4. Just as in the case of a single variable it is sometimes necessary to consider equations in higher dimensions whose coefficients become infinite. There exists no general theory of such problems, but in individual instances it is possible to complete the solution

of the problem and to obtain an expansion in the eigenfunctions of the problem. As an example we consider the equation of a vibrating gas in space

$$\Delta u = u_{tt}.$$

When using Fourier's method to solve this equation, we are led to the problem of determining the eigenvalues and eigenfunctions of the equation

$$\Delta u + \lambda u = 0$$

in some region D. If D is a sphere of radius 1 with center at the origin, then, going over to spherical coordinates and assuming

$$u(\rho, \theta, \phi) = f(\rho)\, Y(\theta, \phi),$$

we obtain for the function $Y(\theta, \phi)$ the equation

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\phi}\frac{1}{\sin\theta}\frac{\partial Y}{\partial\phi}\right] + \frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial Y}{\partial\theta}\right] + kY = 0,$$

whose coefficients become infinite at $\theta = 0$ and $\theta = \pi$. As boundary conditions for this equation we take the continuity and uniqueness of the solution on the sphere $\rho = 1$. Under these conditions we obtain, just as in the case of continuous coefficients, an infinite sequence of eigenvalues $k_n = (n+1)n$. To every eigenvalue k_n there correspond $2n+1$ linearly independent eigenfunctions $Y_n^m(\theta, \phi)$ (spherical functions of order $n, m = 1, 2, \ldots, 2n+1$). The sequence of eigenfunctions turns out to be complete on the surface of the sphere, and every function which is continuous and sufficiently smooth on the sphere can be expanded in a uniformly converging series in the spherical functions.

5. *Variational methods for the approximate determination of eigenfunctions and eigenvalues* (Rayleigh, Ritz, Galyorkin[†]).

As was shown in § 22 the problem of finding the first eigenvalue and eigenfunction of equation (1, 22) under the boundary conditions

$$X(0) = X(l) = 0$$

is equivalent to the problem of finding the minimum of the functional

$$G(X) = \int_0^l (pX'^2 + qX^2)\,dx \tag{9,26}$$

under the condition $$H(X) = \int_0^l \rho X^2 dx = 1 \tag{10,26}$$

[†] Cf. L. V. Kantorovich and V. I. Krylov, *Approximation Methods of Higher Analysis*, Gostekhizdat (1949), pp. 258–373 (in Russian).

in the class of functions $X(x)$ continuous and differentiable on the interval $[0,l]$ and vanishing at its end-points, We consider an arbitrary system of infinitely many linearly independent functions $\phi_n(x)$, $0 \leqslant x \leqslant l$ which are continuously differentiable and satisfy the boundary conditions

$$\phi(0)=\phi(l)=0.$$

We seek an approximate solution of the above extremum problem in the form of a linear combination of a finite number of functions

$$X_N(x)=\sum_{n=1}^{N} a_n \phi_n(x). \tag{11, 26}$$

Substituting (11, 26) in (9, 26) and (10, 26) and integrating, we come to the problem of finding the minimum of the quadratic form

$$\begin{aligned} g(a_1,\dots,a_N) &= \sum_{n,m=1} a_m a_n \int_0^l [p\phi_n'(x)\,\phi_m'(x)+q\phi_n(x)\,\phi_m(x)]\,dx \\ &= \sum_{n,m=1}^{N} A_{mn} a_m a_n \end{aligned}$$

under the condition

$$h(a_1,\dots,a_N)=\sum_{n,m=1}^{N} a_m a_n \int_0^l p\phi_n\phi_m\,dx=\sum_{n,m=1}^{N} B_{mn} a_m a_n = 1.$$

This is a problem of the differential calculus which can be solved easily in practice since the derivatives of g and h with respect to the a_k are linear functions of $a_1,\dots,a_N$ and, consequently, the system of equations

$$\partial(g-\lambda h)/\partial a_k = 0 \quad (k=1,2,\dots,N) \tag{12, 26}$$

is simply a linear homogeneous system of equations with respect to the a_k. The determinant of this system is a polynomial of degree N with respect to λ. This polynomial vanishes for

$$\lambda=\lambda_1^{(N)},\dots,\lambda_N^{(N)}, \quad \lambda_1^{(N)}\leqslant\lambda_2^{(N)}\leqslant\dots\leqslant\lambda_N^{(N)}.$$

All λ's are real. For every $\lambda_i^{(N)}$ there exists a non-trivial solution $a_1^{(i)},\dots,a_N^{(i)}$ of the system (12, 26). If $\lambda_i^{(N)}$ is a root of multiplicity k of the determinant, then the system (12, 26) has, for $\lambda=\lambda_i^{(N)}$, k linearly independent solutions $(a_1^{(i)},\dots,a_N^{(i)})$.

If the system of functions $\phi_n(x)$ is complete in the class of admissible functions, then for every fixed value of i and for $N\to\infty$

$$\lambda_i^{(N)}\to\lambda_i \quad \text{and} \quad X_i^{(N)}(x)\to X_i(x).$$

Here λ_i and $X_i(x)$ are, respectively, the ith eigenvalue and the ith eigenfunction of the given problem. In this case, for values of i not exceeding a certain fixed number M and for N sufficiently large, the numbers $\lambda_i^{(N)}$ are approximations to the first M eigenvalues of equation (1, 22) under the boundary conditions

$$X(0) = X(l) = 0,$$

and the functions
$$X_i^{(N)}(x) = \sum_{n=1}^{N} a_n^{(i)} \phi_n(x)$$

are approximations to the first M eigenfunctions.

How rapidly the $X_i^{(N)}(x)$ converge to $X_i(x)$ depends in an essential manner on the choice of the functions $\phi_n(x)$ and on the degree of smoothness of the coefficients $p(x)$, $q(x)$, $\rho(x)$. For a detailed discussion of this problem see N. M. Krylov and N. N. Bogolyubov.†

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† For relevant literature see *Thirty Years of Mathematics in the U.S.S.R.*, Gostekhizdat (1948), pp. 773–80 (in Russian).

CHAPTER III

ELLIPTIC EQUATIONS

§ 27. Introduction

As the simplest representative of the class of elliptic equations we consider the Laplace equation

$$\partial^2 u/\partial x_1^2 + \ldots + \partial^2 u/\partial x_n^2 = 0. \tag{1,27}$$

The fundamental properties of the solutions of this equation do not depend on n. For simplicity in notation we always consider the case $n = 2$. Unless otherwise stated our considerations are valid for the case $n > 2$. At the end of this chapter we give a survey of results known to hold for more general elliptic equations.

Elliptic equations describe stationary, steady states. For instance, in § 1 we saw that the steady temperature u of a homogeneous plate or body satisfies the Laplace equation and that this equation describes the equilibrium form of a membrane stretched over some space curve. Potentials of gravitational and of stationary electrical fields also satisfy Laplace's equation at points free from masses and electrical charges, respectively.

One of the most fundamental properties of solutions of elliptic equations is their smoothness. (This is in accordance with the fact that elliptic equations describe steady states. It is physically clear that all initial 'rough spots' must 'even out' by the time the steady state is established.) For instance, as will be shown in this chapter, all continuous solutions of Laplace's equation are analytic in all independent variables. It is not true, however, that all solutions of Laplace's equation are analytic. Thus the function

$$u(x, y) = \Re \exp\{-z^{-4}\} \quad \text{for} \quad z \neq 0,\ z = x + iy,$$
$$u(0, 0) = 0$$

satisfies the Laplace equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \tag{2,27}$$

everywhere without being analytic in the neighborhood of the origin. What is more, this function is actually discontinuous at the origin. Thus, in order to be able to claim that the function u is analytic, we must assume that it is of a certain degree of smoothness to begin with. The assumption of the boundedness of u turns out to be sufficient. Continuous solutions of Laplace's equation are called *harmonic* functions.

A typical boundary-value problem for elliptic equations is the *Dirichlet problem* (or first boundary-value problem) mentioned already in § 1. This problem can be stated as follows: *Let f be a continuous function prescribed on the boundary Γ of some finite region G of the space $(x_1, \ldots, x_n)$. We are to find a function $u(x_1, \ldots, x_n)$ which is harmonic in the interior of G and equal to f on Γ. (The term 'equal to f on Γ' means that the function coinciding with u in G and with f on Γ must be continuous in $\bar{G} = G + \Gamma$.)*

If, for example, $u(x, y, z)$ denotes the temperature of a body at the point (x, y, z), then Dirichlet's problem consists in finding the temperature in the interior of the body knowing the temperature at the boundary of the body.

Assuming the existence of a solution of Dirichlet's problem, we prove its uniqueness and its continuous dependence on the boundary function. Next we prove the existence of a solution under very general assumptions about the boundary of the region considered.

§ 28. The minimum-maximum property and its consequences

1. *The maximum-minimum theorem. Consider a harmonic function $u(x, y)$ continuous in some closed bounded region $\bar{G} = G + \Gamma$. Then the values of u in G cannot exceed its maximum on Γ nor can they be less than its minimum on Γ.*

Proof.[†] Let m denote the maximum of u on Γ. Let us assume that, at some points of G, u assumes values greater than m. Then the maximum M of u in $\bar{G}$ must also be greater than m, and it must be assumed at some interior point Q of G. We now move the origin to the point Q. Under this transformation u remains harmonic. Next we construct the auxiliary function

$$v(x, y) = u(x, y) + \frac{M - m}{2d^2}(x^2 + y^2),$$

where d is the diameter of our region. (d is equal to the least upper bound of the distances between pairs of points of G. It follows that $x^2+y^2<d^2$ for (x, y) in G.) It is clear that $v(0,0)=u(0,0)=M$. On the other hand, if a point (x,y) belongs to the boundary Γ of the region G, then

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[†] This proof is due to I. I. Privalov, *Matematichesky sbornik*, vol. XXXII (1926), p. 464 (in Russian).

$$v(x,y)\leqslant m+\tfrac{1}{2}(M-m)=\tfrac{1}{2}(m+M)<M.$$

Consequently, $v(x,y)$, like $u(x,y)$, must attain its maximum at an interior point of G. However, this contradicts the fact that at all points of G

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{M-m}{d^2}=\frac{M-m}{d^2}>0,$$

for at a maximum none of the second derivatives of a function can be positive.

To prove the minimum part of the theorem we apply our result to the function $-u$.

2. The uniqueness of the solution of Dirichlet's problem follows easily from the theorem just proved. In fact, let us assume that two harmonic functions u_1 and u_2 coincide on the boundary of some bounded region G. Then their difference, which is also a harmonic function, is identically equal to zero on the boundary of G. By the preceding theorem we must have

$$u_1-u_2\equiv 0 \quad \text{or} \quad u_1\equiv u_2$$

at all interior points of G.

Another consequence of the maximum-minimum theorem is the *continuous dependence of the solution of Dirichlet's problem on the boundary data* for an arbitrary bounded region G. Thus, let u_1 and u_2 be solutions of Dirichlet's problem for some region G corresponding to prescribed boundary functions f_1 and f_2, respectively. Let $|f_1-f_2|<\epsilon$ on Γ. At all points of the boundary Γ the values of the harmonic function u_1-u_2 which are clearly equal to f_1-f_2 must satisfy the inequality

$$-\epsilon<f_1-f_2<\epsilon.$$

It follows from the maximum-minimum theorem that at all points of G

$$-\epsilon<u_1-u_2<\epsilon,$$

i.e. that $|u_1-u_2|<\epsilon$, which is what we set out to prove.

This result implies the following useful lemma.

LEMMA. *Let there be given a sequence of functions continuous on the closure of a certain bounded region and harmonic in its interior. Let this sequence of functions converge uniformly on the boundary of our region. Then this sequence converges uniformly on the closure of our region.*

Proof. Let $u_1, \dots, u_n, \dots$ be the sequence of functions mentioned in the statement of the lemma. Let f_i denote the values of u_i on the boundary Γ of G. By assumption, the sequence of the f_i converges uniformly on Γ. By Cauchy's criterion, given $\epsilon > 0$, there exists an N such that $|f_n - f_m| < \epsilon$ on Γ, provided that $n, m > N$. It follows from our previous results that for these n and m, $|u_n - u_m| < \epsilon$ in all of $\bar{G}$. In view of the sufficiency of Cauchy's criterion, the sequence $u_1, \dots, u_n, \dots$ converges uniformly in $\bar{G}$.

§ 29. Solution of the Dirichlet problem for a circle

1. Let $f(s)$ be a given function defined and continuous on the circumference of a unit circle. Here s denotes arc length of the circle measured from some fixed point, and it is assumed that $f(0) = f(2\pi)$. We wish to construct a function u harmonic in the interior of our circle and equal to $f(s)$ on its circumference.

We put the origin at the center of the circle and we draw the x-axis through the point $s = 0$. We now go over to polar coordinates and we take the origin as the pole of the polar coordinate system, and the x-axis as its polar axis. Then the equation of the circumference of our circle will be simply $\rho = 1$, and $f(s)$ can be written as $f(\phi)$, where ϕ is the polar angle of a point on the circumference of the unit circle.

To solve our problem we use Fourier's method. We assume that $f(s)$ has a piecewise continuous derivative; an assumption we shall be able to discard later. In polar coordinates Laplace's equation takes the form

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = 0. \tag{1,29}$$

We seek a solution of this equation in the form of a product of two functions:

$$u = R(\rho)\,\Phi(\phi). \tag{2,29}$$

Substituting this value of u in (1, 29) and separating variables, we obtain (in analogy to § 20) the following ordinary differential equations for the functions $R(\rho)$ and $\Phi(\phi)$:

$$\Phi'' + \lambda\Phi = 0, \tag{3,29}$$

$$\rho^2 R'' + \rho R' - \lambda R = 0. \tag{4,29}$$

It follows from our problem that the function $\Phi(\phi)$ must be periodic with period 2π. This can happen only if $\lambda = 0$ or λ equal to a square of an integer. Putting $\lambda = n^2$ and

$$\Phi_n = a_n \cos n\phi + b_n \sin n\phi,$$

we obtain from (4, 29) the following equation for R:

$$\rho^2 R'' + \rho R' - n^2 R = 0.$$

This equation† has the linearly independent solutions $R = \rho^n$ and $R = \rho^{-n}$. Since the second of these solutions is discontinuous at the origin, it follows that the particular continuous solutions of the form (2, 29) in the interior of the unit circle are the functions

$$u_n(\rho, \phi) = \rho^n (a_n \cos n\phi + b_n \sin n\phi).$$

In addition, for $\lambda = 0$, we get the solution $u_0(\rho, \phi) = \text{const.}$ which we denote by $\frac{1}{2}a_0$. For arbitrary bounded a_n and b_n the series

$$u(\rho, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\phi + b_n \sin n\phi) \qquad (5, 29)$$

converges and satisfies equation (1, 29) at any interior point of the circle, since, for $\rho < 1$, this series and its derivatives can be majorized by the convergent series

$$M(1 + \rho_1 + \rho_1^2 + \ldots + \rho_1^n + \ldots), \quad \text{where} \quad \rho < \rho_1 < 1, \qquad (6, 29)$$

and by the convergent series obtained by differentiating (6, 29) with respect to ρ_1. We thus see that the function $u(\rho, \phi)$ defined by the formula (5, 29) is a harmonic function for arbitrary bounded a_n and b_n, provided that $\rho < 1$.

Putting $\rho = 1$ and $u(1, \phi) = f(\phi)$, we obtain from (5, 29) the equality

$$f(\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi).$$

In view of the assumptions made about $f(\phi)$ this equality will hold if we define a_0, a_n and b_n as the Fourier coefficients of $f(\phi)$:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\phi)\, d\phi, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \cos n\phi\, d\phi,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin n\phi\, d\phi. \qquad (7, 29)$$

For the series (5, 29) with coefficients defined by the formulas (7, 29) to be the solution of Dirichlet's problem it must still be shown that this function (i.e. (5, 29)) is continuous in the closed unit circle $\rho \leqslant 1$ (cf. § 27). However, for $\rho \leqslant 1$, we can majorize the series (5, 29) by means of the series

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† The substitution $\rho = \exp\{t\}$ takes this equation into one with constant coefficients.

$$\tfrac{1}{2}|a_0| + \sum_{n=1}^{\infty} |a_n| + |b_n|,$$

which converges in view of the assumption that $f(\phi)$ is continuous and piecewise smooth.†

We have thus shown that the series (5, 29) is the solution of Dirichlet's problem for the unit circle for continuous and piecewise smooth boundary functions.

2. We now transform the series (5, 29) by replacing a_n and b_n by their respective values, as given by (7, 29). We get

$$\begin{aligned} u(\rho,\phi) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\phi + b_n \sin n\phi) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,d\psi + \frac{1}{\pi}\sum_{n=1}^{\infty} \rho^n \left\{ \int_0^{2\pi} f(\psi)\cos n\psi\,d\psi \cos n\phi \right. \\ &\qquad \left. + \int_0^{2\pi} f(\psi)\sin n\psi\,d\psi \sin n\phi \right\} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,d\psi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(\psi)\rho^n \cos n(\psi-\phi)\,d\psi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\left(1 + 2\sum_{n=1}^{\infty}\rho^n \cos n(\psi-\phi)\right)d\psi. \end{aligned}$$

We transform the expression in brackets by putting

$$\psi - \phi = \omega \quad \text{and} \quad z = \rho \exp\{i\omega\}.$$

We then get

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty} \rho^n \cos n\omega &= -1 + 2\sum_{n=0}^{\infty}\rho^n \cos n\omega \\ &= -1 + 2\Re \sum_{n=0}^{\infty} \rho^n \exp\{in\omega\} = -1 + 2\Re\,(1-\rho\exp\{i\omega\})^{-1} \\ &= 1-\rho^2/1+\rho^2-2\cos\omega. \end{aligned} \qquad (8, 29)$$

Hence, for $\rho < 1$,

$$u(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\psi-\phi)}\,d\psi. \qquad (9, 29)$$

The integral (9, 29) is called *Poisson's integral.*

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† R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. I (1953), Interscience, New York–London. For a Russian source see I. I. Privalov, *Fourier Series*, GTTI (1937).

To obtain the solution of Dirichlet's problem for a circle about the origin of radius R we need only replace ρ in formula (9, 29) by ρ/R. As the variable of integration we can take, instead of ψ, $s=R\psi$. We then get Poisson's integral for an arbitrary circle:

$$u(\rho, \phi)=\frac{1}{2\pi R}\int_0^{2\pi R} f(s)\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\psi-\phi)}ds. \quad (10, 29)$$

3. We have thus shown that formula (10, 29) gives the solution of Dirichlet's problem for the circle provided that the function $f(s)$ prescribed on the boundary of the circle in addition to being continuous satisfies additional smoothness requirements. We now show that these additional requirements are superfluous. We assume only that $f(s)$ is continuous on the circumference of the circle considered, and we show that the function given by formula (10, 29) represents in the interior of the circle a harmonic function which is equal to $f(s)$ on the boundary of the circle. To prove the harmonicity property of $u(\rho, \phi)$, we observe that, for $\rho<R$, it is possible to differentiate this function under the integral sign with respect to both ρ and ϕ (and, consequently, also with respect to x and y) any number of times, for the resulting function under the integral sign is always continuous. Differentiation of the function

$$R^2-\rho^2/R^2+\rho^2-2R\rho\cos(\psi-\phi)$$

shows that its Laplacian vanishes. The latter fact can also be proved indirectly. Thus, let us assume that the Laplacian of the above function does not vanish for $\psi=\psi_0$, $\rho=\rho_0$ and $\phi=\phi_0$. Then it is of the same sign (+ or −) in some neighborhood of (ψ_0, ρ_0, ϕ_0). It would then be possible to choose an arbitrarily smooth function $f_0(s)$ which would vanish everywhere outside an arbitrarily small neighborhood of $s_0=R\psi_0$ and would be of fixed sign in that neighborhood. But then the function

$$u(\rho, \phi)=\frac{1}{2\pi R}\int_0^{2\pi R} f_0(s)\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\psi-\phi)}ds$$

would not be harmonic in the neighborhood of the point (ρ_0, ϕ_0), which contradicts the result proved before. We have thus shown that for an arbitrary continuous function $f(s)$ Poisson's integral represents a function which is harmonic in the interior of the circle considered.

It remains to show that this function takes on the prescribed values on the circumference, i.e. that the function $u(\rho, \phi)$, defined by the equality (10, 29) for $\rho < R$ and by the equality

$$u(R, \phi) = f(s) \quad (s = R\phi)$$

is continuous on the closure $\bar{K}$ of the circle K under consideration. To this end we construct a sequence of functions $f_n(s)$ satisfying the smoothness conditions formulated in the beginning of this section, which converges uniformly to the function $f(s)$ prescribed on the circumference. According to the lemma at the end of § 28, the sequence of harmonic functions

$$u_n(\rho, \phi) = \frac{1}{2\pi R} \int_0^{2\pi R} f_n(s) \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos(\psi - \phi)} ds$$

converges uniformly in $\bar{K}$. Consequently, the limit function

$$u(\rho, \phi) = \frac{1}{2\pi R} \int_0^{2\pi R} f(s) \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos(\psi - \phi)} ds$$

is continuous on $\bar{K}$, which is what we wished to prove.

Remark. The solution of Dirichlet's problem for an n-dimensional sphere is given by a formula analogous to formula (10, 29). For $n = 3$, the corresponding formula has the form

$$u(r, \theta, \phi) = \frac{1}{4\pi R^2} \iint_{\Sigma} f(\theta', \phi') \frac{R^2 - r^2}{(R^2 - 2rR \cos\gamma + r^2)^{\frac{3}{2}}} d\sigma.$$

Here integration extends over the surface Σ of the sphere of radius R, and γ denotes the angle between the radius vector (r, θ, ϕ) and the variable radius vector (R, θ', ϕ') on the surface of the sphere.

§ 30. Theorems on the fundamental properties of harmonic functions

The proofs of virtually all of these theorems are based on the fact that, if a function u is harmonic on a closed circle $\bar{K}$, then it is possible to represent it on this circle in the form of a Poisson integral which can be easily investigated. In fact, if the function u is harmonic and, consequently, continuous on $\bar{K}$, it is possible to construct a function u_1 in the form of a Poisson integral such that u_1 is harmonic in K and equal to u on the boundary of K. But in view of the uniqueness of the solution of Dirichlet's problem we must have $u \equiv u_1$, i.e. the Poisson integral represents the original function u.

THEOREM 1 (*on the arithmetic mean*). *Let the function u be harmonic in some circle K and continuous on the closure $\bar{K}$ of K. Then the value of u at the center of K is equal to the arithmetic mean of its values on the circumference of K.*

Proof. We represent u in K by means of formula (10, 29). For $\rho = 0$ we obtain

$$u(0, \phi) = \frac{1}{2\pi R}\int_0^{2\pi R} f(s)\,ds = \frac{1}{2\pi R}\int_0^{2\pi R} u(R, \psi)\,ds,$$

where $\psi = s/R$. But this means that $u(0, \phi)$ is equal to the arithmetic mean of the values of u on the circumference of the circle K of radius R.

THEOREM 2. *Every harmonic function $u(x, y)$ is analytic in x and y.*

This means that if the point (x_0, y_0) is an interior point of the region of harmonicity of $u(x, y)$, then it is possible to represent $u(x, y)$ in a neighborhood of (x_0, y_0) in a power series in $(x - x_0)$, $(y - y_0)$.

Proof. We represent $u(x, y)$ as a Poisson integral in some circle K with center (x_0, y_0). For simplicity in notation we assume $x_0 = y_0 = 0$. According to formula (8, 29), with ρ replaced by ρ/R,

$$\frac{(R^2-\rho^2)\,d\psi}{R^2+\rho^2-2R\rho\cos(\phi-\psi)} = 2\Re\left[-\frac{1}{2} + \frac{R}{R-\rho\exp\{i(\phi-\psi)\}}\right]d\psi$$

$$= -d\psi + 2\Re\frac{iR\exp\{i\psi\}\,d\psi}{i(R\exp\{i\psi\}-\rho\exp\{i\phi\})} = -d\psi + 2\Re\frac{d\alpha}{i(\alpha - z)}. \quad (1, 30)$$

Here $z = x + iy = \rho\exp\{i\psi\}$, $\alpha = R\exp\{i\psi\}$, so that α is always a boundary point of K. We multiply the equality (1, 30) by $(\frac{1}{2}\pi)f(s)$, where $f(s)$ gives the values of u on the boundary of K, and integrate over the boundary of K. In view of (8, 29), (10, 29) and (1, 30), we get for an interior point (x, y) of K

$$u(x, y) = -\frac{1}{2\pi R}\int_0^{2\pi R} f(s)\,ds + \Re\frac{1}{\pi R i}\oint\frac{f(\alpha)\,d\alpha}{a - z}, \quad (2, 30)$$

where $\oint$ denotes integration over the circumference of K. The first integral on the right represents a constant equal to $-u(0, 0)$. As for the second integral, the theory of analytic functions tells us that it represents an analytic function of $z = x + iy$ in K which can be expanded in K in powers of z. Now, $u(x, y)$ is the real part of a power series in z. This implies that if K_1 is any circle concentric with and smaller than K, then $u(x, y)$ can be represented in K_1 by a uniformly converging series in powers of x and y.

THEOREM 3. (*Strong form of the maximum-minimum theorem.*) *Let the function u be harmonic in the interior of a region G. Let A be an interior point of G at which the value of u is equal to the least upper bound (greatest lower bound) of its values in $\bar{G}$. Then u is constant.*

It is clear that if the region G is finite and if it is possible to define u in $\bar{G}$ so that the resulting function (which we shall also denote by u) is continuous in $\bar{G}$, then, clearly, the least upper bound (greatest lower bound) of the values of u in $\bar{G}$ is equal to its maximum (minimum).

Proof. Let the value of u at an interior point A of G be equal to the least upper bound, say, of its values on $\bar{G}$. Now let A be the center of some circle K contained entirely in the interior of G. Since the values of u on the boundary of K cannot exceed the value $u(A)$, u must, by the theorem of the arithmetic mean, be identically equal to $u(A)$ at all points of the boundary of K. Hence, u must be equal to $u(A)$ in the interior of K. Now let A_1 be some point in K or on its boundary and let K_1 be a circle in the interior of G with center at A_1. Then u must be equal to $u(A)$ in K_1. To show that $u(B)=u(A)$ for an arbitrary interior point B of G we join A and B by means of a polygonal line AB consisting of a finite number of segments and contained in the interior of G. It is possible to construct a finite chain of circles in G covering AB such that the center of each of the covering circles lies in the interior or on the boundary of its predecessor. Denoting by K_1 the first of these circles, by K_2 the second, etc., we can show, using the technique described above, that $u(B)=u(A)$.

THEOREM 4. (*Harnack's first theorem.*) *Let $u_k(x,y)$ $(k=1,2,\dots)$ be a sequence of functions harmonic in the interior of a finite region G and continuous in $\bar{G}$. If this sequence converges uniformly on the boundary of G, then it also converges uniformly in the interior of G to a limit function which is harmonic in the interior of G.*

Proof. According to the lemma formulated at the end of § 28, the sequence of functions $u_n(x,y)$ converges uniformly in the interior of G. It remains to show that the limit function is harmonic in the interior of G. To this end we choose a circle K in the interior of G with center at some arbitrary interior point Q of G. We now represent each of the functions u_n in K as a Poisson integral. Thus, let

$$u_n(x,y)=\frac{1}{2\pi}\int_0^{2\pi} f_n(\psi)\frac{R^2-\rho^2}{R^2+\rho^2-2\rho R\cos(\phi-\psi)}\,d\psi, \qquad (3,30)$$

where $f_n(\psi)$ are the values of u_n on the circumference of the circle K of radius R. In view of the uniform convergence of the sequence of functions $f_n(\psi)$ and of the convergence of the u_n at an arbitrary interior point (x, y) of K, we may go over to the limit on both sides of equation (3, 30). Denoting the limit functions by $u(x, y)$ and $f(\psi)$, respectively, we obtain

$$u(x,y)=\frac{1}{2\pi}\int_0^{2\pi} f(\psi)\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-\psi)}\,d\psi.$$

This implies the harmonicity of $u(x, y)$ in K.

THEOREM 5. (*Harnack's second theorem.*) *Let a series of non-negative functions $u_n(x, y)$ harmonic in the interior of G converge at some interior point A of G. Then this series converges to a harmonic function in all of G, and the convergence is uniform in every closed bounded region in G.*

Proof. We first show that our series converges uniformly in every circle K_1 of radius R with center A, provided the closure $\bar{K}_1$ of K_1 is contained in G. To this end we pick a circle K^* concentric with K_1 and of radius $R+\epsilon$, where ϵ is so small that $\overline{K^*}$ is still in G. We represent each of the functions u_n in K^* as a Poisson integral:

$$u_n(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi} u_n(R+\epsilon,\psi)\frac{(R+\epsilon)^2-\rho^2}{(R+\epsilon)^2+\rho^2-2(R+\epsilon)\rho\cos(\phi-\psi)}\,d\psi. \tag{4, 30}$$

Since $-1\leqslant\cos(\phi-\psi)\leqslant+1$, it follows that

$$\frac{R+\epsilon-\rho}{R+\epsilon+\rho}\leqslant\frac{(R+\epsilon)^2-\rho^2}{(R+\epsilon)^2+\rho^2-2(R+\epsilon)\rho\cos(\phi-\psi)}\leqslant\frac{R+\epsilon+\rho}{R+\epsilon-\rho}. \tag{5, 30}$$

Making use of the fact that $u_n(R+\epsilon,\ \psi)\geqslant 0$, we get, in view of (4, 30) and (5, 30),

$$\frac{1}{2\pi}\frac{R+\epsilon-\rho}{R+\epsilon+\rho}\int_0^{2\pi} u_n(R+\epsilon,\psi)\,d\psi\leqslant u_n(\rho,\phi)$$
$$\leqslant\frac{1}{2\pi}\frac{R+\epsilon+\rho}{R+\epsilon-\rho}\int_0^{2\pi} u_n(R+\epsilon,\psi)\,d\psi.$$

Now, by the theorem on the arithmetic mean,

$$\frac{1}{2\pi}\int^{2\pi} u_n(R+\epsilon,\psi)\,d\psi=u_n(0,\psi)=u_n(A).$$

Consequently,

$$\frac{R+\epsilon-\rho}{R+\epsilon+\rho}u_n(A) \leqslant u_n(\rho,\phi) \leqslant \frac{R+\epsilon+\rho}{R+\epsilon-\rho}u_n(A). \tag{6,30}$$

It is now clear that, if the sequence of the u_n converges at A, it also converges in $\bar{K}_1$. It follows by Harnack's first theorem that the limit function is harmonic in the interior of K_1.

To prove the convergence of the sequence of the u_n at an arbitrary point B of G, we join A to B by means of a polygonal line l consisting of a finite number of segments and lying in G. That this is possible follows from the definition of a region. The polygonal line l (together with its end-points A and B) forms a closed set. Since l has no points in common with the boundary of G, it must be a positive distance δ away from the closed set which is the boundary of G. We now choose a point A_2 in the intersection of the line l and the circumference of K_1. About A_2 as center we draw a circle K_2 of radius $\frac{1}{2}\delta$. According to what has been said before, the sequence of the u_n converges uniformly in $\bar{K}_2$. Likewise, it converges uniformly in $\bar{K}_3$, where K_3 is a circle of radius $\frac{1}{2}\delta$ with center belonging to the intersection of l and the circumference of K_2. It is possible to cover l by means of a finite number of such circles $K_i\,(i=1,\ldots,N)$, and the covering can be effected so that B is in the interior of K_N. This implies the convergence of the sequence of the u_n at all points of the polygonal line l and, in particular, at the point B. Since the convergence is uniform in each of the circles K_i and, in particular, in K_N, it follows, by Harnack's first theorem, that the limit function is harmonic in the neighborhood of B.

We now show that the sequence of the $u_n(x, y)$ converges uniformly in every closed and bounded set F in the interior of G. By the Heine-Borel theorem, it is possible to cover F by means of a finite number of circles $K_1,\ldots,K_N$ contained with their closures in the interior of G. According to what has been proved in the preceding paragraph, the sequence of the $u_n(x,y)$ converges at the center of each of these circles. Consequently, according to what has been proved above, this sequence converges uniformly in each of the circles K_i and so, in all of F.

THEOREM 6. (*Liouville's theorem.*) *A non-constant function harmonic in the whole plane cannot be bounded from above or from below.*

Proof. Let $u(x,y) \geqslant M$, say, for all (x,y) and for some constant M. By adding, if necessary, some constant to $u(x,y)$ we can always take $M \geqslant 0$. Under this assumption we show that the value of u at an arbitrary point $Q(\rho, \phi)$ is equal to its value at the origin O. This will prove that u is a constant. Choosing a circle K about O with radius R large enough for $Q(\rho, \phi)$ to be in K and representing u in K as a Poisson integral, we get

$$u(Q) = \frac{1}{2\pi}\int_0^{2\pi} u(R, \psi)\frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\phi - \psi)}d\psi.$$

Whence, as in (6, 30), we get

$$\frac{R-\rho}{R+\rho}u(O) \leqslant u(Q) \leqslant \frac{R+\rho}{R-\rho}u(O).$$

For $R \to \infty$ we have $u(O) \leqslant u(Q) \leqslant u(O)$. Hence $u(Q) = u(O)$.

THEOREM 7. (*On removable singularities.*) *Let $u(x, y)$ be harmonic and bounded in a deleted neighborhood of a point A at which the function is not defined. We claim that it is possible to define $u(A)$ so that u is harmonic in the above neighborhood of A including the point A itself.*

Proof. We assume for the sake of simplicity that A is the origin of our coordinate system. Let K be a circle about A of radius R contained entirely in the neighborhood of A under consideration. Let u_1 be a function harmonic in the interior of K and equal to u on its boundary. We put $u - u_1 = v$. The function $v(x, y)$ is harmonic and bounded in K except for the point A at which it is not defined. We now show that $v \equiv 0$, i.e. $u = u_1$ in K except at the point A at which it is not defined. Assuming this to have been proved, we can prove our theorem by putting $v(A) = 0$, so that $u = u_1$.

To show that $v \equiv 0$ in the circle K minus the point A, we construct in K the function

$$w_\epsilon(P) = M\left(\ln\frac{\rho}{R}\Big/\ln\frac{\epsilon}{R}\right),$$

where M is the least upper bound of $|v|$ in K, ϵ is a small positive number, and $\rho = AP$ (i.e., the distance from A to P). The function $w_\epsilon(P)$ vanishes on the boundary of the circle K, and it is equal to M on the boundary of the circle K_ϵ concentric with K and having the radius ϵ. It follows from the maximum theorem for harmonic functions that

$$|v(P)| \leqslant M\left(\ln\frac{AP}{R}\Big/\ln\frac{\epsilon}{R}\right)$$

for an arbitrary point P of the ring bounded by the circumferences of K and K_ϵ, and for an arbitrary ϵ. If now $\epsilon \to 0$, the right side of the above inequality goes to zero. Since the left side of this inequality is independent of ϵ, its value must be zero.

Remark. All the properties of harmonic functions in two variables proved in this section carry over to harmonic function in any number of independent variables. The proofs are analogous to those given for the two-dimensional case.

§ 31. Proof of the existence of a solution of Dirichlet's problem

The idea of the proof given below originated with Poincaré. The proof of Poincaré was improved somewhat by Perron. Since the following considerations are applicable to regions of an arbitrary number of dimensions, we shall conduct them in n-dimensional space.

1. *Basic definitions and the method of solution of the problem.* Let v be a given continuous function defined in an n-dimensional bounded region G and on its boundary. By K we always denote some n-dimensional sphere (for $n = 2$, K is, of course, a circle) all of whose interior points belong to G, and by $(v)_k$, a continuous function equal to v outside and on the boundary of K and harmonic in the interior of K. It is clear that for the function v to be harmonic it is necessary and sufficient that $(v)_k \equiv v$ for every sphere K.

We shall call the function v *superharmonic* (*subharmonic*) if for every sphere K

$$(v)_k \leqslant v \quad ((v)_k \geqslant v). \tag{1,31}$$

Let f be a continuous function defined on the boundary of G. Let the function v be superharmonic (subharmonic) in G. Finally, let

$$v \geqslant f \quad (v \leqslant f)$$

on the boundary of G. Then v is said to be an *upper* (*lower*) function for f. In the sequel we shall consider only those superharmonic and subharmonic, upper and lower functions which are continuous in the closure $\bar{G}$ of G.

The Poincaré-Perron method consists in the following. We determine the family of all upper functions for a given bounded region G and for a given continuous function f defined on the boundary of G. It is clear that this family is not empty since every constant $c \geqslant \sup f$ is already an upper function. We now define a function u in $\bar{G}$ by putting $u(P)$ equal to the greatest lower bound of the values at P of all upper functions. We then prove that u is harmonic in the interior of G and that it assumes the prescribed boundary values at all those boundary points of G which satisfy certain conditions to be discussed later. As a preliminary we prove some properties of superharmonic and upper functions.

2. *Certain properties of superharmonic and upper functions.*

THEOREM 1. (*a*) *Every harmonic function is superharmonic and subharmonic.*

(*b*) *If v is superharmonic and u is harmonic, then $v \pm u$ is superharmonic.*

(*c*) *The sum of two (and, consequently, of an arbitrary finite number) superharmonic functions is superharmonic.*

(*d*) *If v is superharmonic and w is subharmonic, then $v-w$ is superharmonic.*

Analogous theorems hold for subharmonic functions.

The first of these statements is obvious. The remaining statements follow easily from the relation

$$(v_1+v_2)_k = (v_1)_k + (v_2)_k.$$

Using this relation we prove, for example, the statement (*c*). Thus, let v_1 and v_2 be two superharmonic functions. Then

$$(v_1)_k \leqslant v_1, \quad (v_2)_k \leqslant v_2.$$

Consequently, $$(v_1+v_2)_k = (v_1)_k + (v_2)_k \leqslant v_1 + v_2,$$

i.e. the function $v_1 + v_2$ is superharmonic.

THEOREM 2. *A function v superharmonic in the region G assumes its least value on the boundary of G.*

Proof. Let us assume that the function v assumes its least value m at some point P in the interior of G. It must then be possible to draw about P a sphere K touching the boundary of G such that v is identically equal to m on the boundary of K; otherwise, at P

$$v < (v)_k.$$

It follows that the function v must be equal to m at some boundary point of G.

THEOREM 3. *Every upper function v is either greater than or equal to every lower function w.*

Proof. The superharmonic function $v-w$ assumes its least value on the boundary of the region (cf. Theorem 2) where it is non-negative. Consequently, this function is non-negative in the interior of the region.

THEOREM 4. *The function*

$$v = \min\{v_1, v_2, \dots, v_n\},$$

where $v_1, v_2, \dots, v_n$ are upper functions, is also an upper function.

Proof. It is clear that the function v is continuous in $\bar{G}$ and that on the boundary of G

$$v \geqslant f.$$

It remains to show that v satisfies the inequality (1, 31) for every sphere K. To this end we note that $v(P)$ is equal to the value at P of one of the functions $v_1, \dots, v_n$, say, v_1. Hence

$$v = v_1 \geqslant (v_1)_k \geqslant (v)_k$$

at P, which proves the inequality (1, 31). Here we have made use of the fact that, if $v \leqslant v_1$, then $(v)_k$ is always $\leqslant (v_1)_k$.

THEOREM 5. *If v is an upper function, then $(v)_k$ is also an upper function.*

Proof. We put

$$(v)_k = z.$$

It is clear that the only property of upper functions to be proved in the case of $(v)_k$ is the property

$$(z)_{k_1} \leqslant z \qquad (2, 31)$$

for every sphere K_1. This property, too, is obvious if the sphere K_1 is contained entirely in the interior or in the exterior of K. It remains to consider the case when the sphere K lies in the interior of K_1 and the case when the boundaries of these spheres intersect.

On the boundary of K_1,

$$z \leqslant v.$$

Hence, in the interior of K_1,

$$(z)_{k_1} \leqslant (v)_{k_1},$$

for the functions $(z)_{k_1}$ and $(v)_{k_1}$ are both harmonic there. In view of the superharmonicity of v,

$$(v)_{k_1} \leqslant v.$$

The functions z and v coincide outside the sphere K and on its boundary. Hence the relation (2, 31) holds outside K and on its boundary when K lies in the interior of K_1 as well as when their boundaries intersect. The functions z and $(z)_{k_1}$ are harmonic in the interior of the intersection $K \cap K_1$. Consequently, if the relation (2, 31) holds on the boundary of the region $K \cap K_1$, it also holds in its interior.

3. *Proof of the harmonicity of u, the greatest lower bound of all upper functions.* To prove the harmonicity of u in the whole region it clearly suffices to prove its harmonicity in every sphere K. We take an upper function v_1 whose value at the center P of the sphere K does not exceed $u(P)+\epsilon$. We may assume v_1 to be harmonic in the interior of K. Otherwise, we could take instead of v_1 the function $(v_1)_k$ which, by Theorem 5, is also an upper function and whose value at P also does not exceed $u(P)+\epsilon$.

We take next an upper function v_2' whose value at P does not exceed $u(P)+\frac{1}{2}\epsilon$. We put

$$v_2=(\min(v_1,v_2'))_k.$$

By Theorems 4 and 5, the function v_2 is also an upper function.

Continuing in this manner we obtain an infinite decreasing sequence of upper functions $v_1, v_2, \ldots, v_n, \ldots$, harmonic in the interior of K. This sequence is bounded from below (Theorem 3). Consequently, according to Harnack's second theorem, this sequence converges uniformly in K to some harmonic function v.

We prove that in K

$$v=u.$$

Suppose that this statement is false. Then there exists an upper function z whose value at some interior point P_1 of the sphere K is less than $v(P_1)$. About P as center we describe a sphere K_1 of radius ρ such that P_1 is on the surface of K_1. Then every function

$$z_n=(\min(z,v_n))_{k_1}$$

is an upper function. Now, since the sequence of the v_n in $\bar{K}_1$ converges uniformly to v, the sequence of the z_n also converges uniformly in $\bar{K}_1$. Consequently, for n sufficiently large, z_n differs by an arbitrarily small amount from the value of $(\min(z,v))_{k_1}$ at P. This latter value is less than $v(P)=u(P)$. This, however, contradicts the assumption that $u(P)$ is the greatest lower bound of the values of all upper functions at the point P.

4. *The behavior of the function u on the boundary of G.*

THEOREM. *The function u takes on the value $f(Q)$ at a boundary point Q of G provided that the following condition holds:*

Condition A. There exists a superharmonic function (barrier) ω_Q which is continuous and single-valued in G and on its boundary and has the following properties:

(1) $\omega_Q(Q)=0$,

(2) *At all points P of $\bar{G}$ other than Q, $\omega_Q(P)>0$.*

Proof. Regardless of how small a positive number ϵ we choose, it is always possible to find a neighborhood U_Q of Q so small that at each of its points P belonging to the boundary of G the following inequality, based on the continuity of f, holds:

$$f(Q)-\epsilon \leqslant f(P) \leqslant f(Q)+\epsilon.$$

Therefore, making use of the fact that the function $\omega_Q(P)$ exceeds some positive constant in the part of $\bar{G}$ outside U_Q, it is easy to prove that for sufficiently large $C>0$ the function

$$\phi(P)=f(Q)-\epsilon-C\omega_Q(P)$$

is a lower function and the function

$$\psi(P)=f(Q)+\epsilon+C\omega_Q(P)$$

is an upper function.

We prove the second of these two statements. It is evident that $\psi(P)$ is superharmonic for every non-negative C. It remains to show that $\psi \geqslant f$ at all boundary points of G. In view of the continuity of f at the point Q, the correctness of this statement for some neighborhood of Q follows directly from the fact that $\epsilon>0$ and $C\omega_Q(P)\geqslant 0$. As for the exterior of this neighborhood, $\omega_Q(P)$ exceeds, by assumption, some positive constant, so that, for C sufficiently large, $C\omega_Q(P)$ can be made as large as we please at all points of the boundary of G outside U_Q.

It is clear that for $\epsilon>0$ the values $u(P)$ of the function u are contained between the values $\phi(P)$ and $\psi(P)$ of the two continuous functions ϕ and ψ. Consequently,

$$f(Q)-\epsilon=\phi(Q)\leqslant u(Q)\leqslant \psi(Q)=f(Q)+\epsilon.$$

Now, since ϵ is arbitrarily small, it follows that

$$u(Q)=f(Q),$$

and the constructed function $u(P)$ is continuous at the point Q.

For $n>2$ it is easiest to construct a barrier at a point Q for which there exists an n-dimensional sphere with center at some point O which touches the boundary of G at the single point Q and whose interior contains no interior points of G. We can then take for the function $\omega_Q(P)$ the function

$$1/\overline{OQ}^{n-2}-1/\overline{PO}^{n-2},$$

where $\overline{PO}$ ($\overline{OQ}$) denotes the distance between the points P and O (O and Q). It is clear that for $n>2$ this function is harmonic.

For $n=2$ it is possible to show that every boundary point Q of a region G bounded by a single simple curve satisfies condition A. Thus, if the point Q is taken as the origin, the function $-p/(p^2+q^2)$, where p is the real and q the imaginary part of $\ln(x+iy/2D)$ ($D=$ the diameter of the region G), possesses all the properties of the function ω_Q. However, the function $-p/(p^2+q^2)$ may not have all these properties if the region G is not simply connected. Thus, for example, the function $-p/(p^2+q^2)$ ceases to be single-valued if G is the region enclosed by the circumferences of two concentric circles and the point Q lies on the smaller of the two circumferences. It is therefore expedient to replace condition A by the following, more general, condition:

Condition B. In some arbitrarily small neighborhood U_Q *of* Q (here U_Q denotes that part of the full neighborhood of Q which belongs to $\bar{G}$) *there exists a single-valued superharmonic function* Ω_Q *(barrier) possessing the following properties:*

(1) Ω_Q *is defined and continuous in* U_Q *and on its boundary.*

(2) $\Omega_Q(Q)=0$.

(3) $\Omega_Q>0$ *at all points other than* Q.

These three properties of Ω_Q imply its fourth property:

(4) $\Omega_Q \geqslant k>0$ *at those boundary points of* U_Q *which belong to* G. *Here* k *denotes a constant.*

We now show that if the point Q satisfies condition B it also satisfies condition A. To this end we construct a function $\omega_Q(P)$ defined as follows:

$$\omega_Q(P)=\min\left\{\frac{2}{k}\,\Omega_Q(P),\,1\right\} \text{ in } U_Q,$$

$$\omega_Q(P)=1 \text{ outside } U_Q.$$

We claim that this function possesses all the properties listed in condition A. In fact,

(1) $\omega_Q(P)$ is continuous in $\bar{G}$.

(2) $\omega_Q(Q)=0$,

(3) $\omega_Q>0$ at all points of $\bar{G}$ except Q.

(4) It remains to show that $\omega_Q(P)$ is superharmonic, i.e., that

$$(\omega_Q)_k \leqslant \omega_Q. \tag{3,31}$$

We denote the part of G where $\omega_Q=1$ by G_1 and the rest of G by G_0. Then (3, 31) will obviously be true if the interior of the sphere K contains only points of G_1 or only points of G_0. There remains the case when the interior of K contains points of G_1 as well as points of G_0. In this case, we first consider the part of the sphere K which belongs to G_1. In this part of K, $\omega_Q = 1$ and $(\omega_Q)_k \leqslant 1$, which implies the validity of (3, 31). To see that relation (3, 31) holds for the points of the intersection $K \cap G_0$ of the regions K and G_0, we note that the function ω_Q is superharmonic and the function $(\omega_Q)_k$ is harmonic in $K \cap G_1$ and in $K \cap G_2$. Also, the values of ω_Q on the boundaries of these regions are not less than the values of $(\omega_Q)_k$.

In the sequel we assume that $n \geqslant 2$, for the boundary-value problem under consideration is trivial if $n = 1$.

It is easy to show that if $n = 2$, then every boundary point Q of G satisfies condition B, provided that there exists a neighborhood U_Q of Q with the following property; it is not possible to draw in U_Q a closed curve surrounding Q and consisting of interior points of G only. In fact, let us put the origin at Q and let us assume that the neighborhood U_Q is so small that the distance between any of its points and the point Q is less than c, where $c < 1$. If we now put $\ln(x+iy) = p+iq$, then the function

$$\Omega_Q = -p/(p^2+q^2)$$

will have all the properties listed in condition B.

If $n \geqslant 2$, it is easy to construct a function Ω_Q for every boundary point Q which can serve as vertex for a circular n-dimensional cone C_Q with straight lines as generators, all of whose points which are sufficiently close to Q lie in the exterior of G. To this end we consider a simply connected region G^* made up of points interior to some n-dimensional sphere S of radius R and center Q and exterior to the cone C_Q. On the boundary of G^* we prescribe a function f^* by putting

$$f^*(P) = \overline{QP},$$

where $\overline{QP}$ denotes the distance between the points Q and P. The greatest lower bound u^* of all upper functions for the region G^* and the function f^* assumes, in view of the criterion formulated on p. 185, the values f^* at all points of the boundary of G^* other than Q at which this criterion cannot be applied. To convince ourselves that the function u^* possesses all the properties of Ω_Q it is necessary to show that u^* vanishes at Q. We observe that $\underline{u}^*(Q) \geqslant 0$, since a function which vanishes identically is a lower function. Here

$$\underline{u}^*(Q) = \underset{\substack{\epsilon \to 0 \\ PQ < \epsilon}}{\underline{\lim}}\, u^*(P), \quad \overline{u}^*(Q) = \underset{\substack{\epsilon \to 0 \\ PQ < \epsilon}}{\overline{\lim}}\, u^*(P).$$

It remains to show that $\bar{u}^*(Q) = 0$.

Let us assume that
$$\bar{u}^*(Q) = c > 0. \tag{4, 31}$$

We take the point Q as the origin and we consider the function

$$u^{**}(x_1, \ldots, x_n) = u^*(kx_1, \ldots, kx_n) \quad (k > 1).$$

Clearly,
$$\bar{u}^{**}(Q) = \bar{u}^*(Q) = c > 0.$$

On the other hand, it is easy to see that at all boundary points other than Q of the region G^{**} in which the function u^{**} is defined we have
$$u^* \leqslant c_1 u^{**}, \tag{5, 31}$$

where c_1 is a constant < 1 depending on k. This means that the relation (5, 31) holds in the interior of G^{**}, for the functions u^* and u^{**} are bounded and harmonic in the interior of G^{**} and their values at the one boundary point Q do not influence their behavior in the interior of G^{**}. This is proved in exactly the same way in which we proved the theorem on removable singularities of harmonic functions in the preceding section. We take the point Q as the center of a sphere K of radius so large that the whole region G^{**} lies in the interior of K. Just as in the case of the theorem on removable singularities we construct a function u_ϵ which vanishes on the surface of K and takes on the value M on the surface of the sphere K_ϵ of radius ϵ, concentric with K. Here M is the least upper bound of $u^* - c_1 u^{**}$. If now P is a point of the part of G^{**} between the spheres K and K_ϵ, then

$$u^* - c_1 u^{**} \leqslant u_\epsilon.$$

However, at every fixed point P the positive quantity $u_\epsilon(P)$ is arbitrarily small for a sufficiently small value of ϵ. Consequently, at every point P of the region G^{**},

$$u^* \leqslant c_1 u^{**},$$

which is what we set out to prove.

The fact that the relation (5, 31) holds at all points of the region G^{**} implies that
$$\bar{u}^*(Q) \leqslant c_1 \bar{u}^{**}(Q) = c_1 c.$$

Since $c_1 < 1$, this relation contradicts (4, 31) for $c > 0$.

The boundary points of G for which condition B holds are called *regular*.

Problem. Show that there exists no function harmonic in a circle minus its center which vanishes on its circumference and takes on the value 1 at its center.

§ 32. The exterior Dirichlet problem

By the exterior Dirichlet problem we mean the following problem.

Let there be given a bounded simply connected region G. Let f be a continuous function prescribed on the boundary of G. We wish to find a function u which is harmonic in the exterior of G and which takes on prescribed boundary values f on the boundary of G.

We say that u takes on prescribed boundary values f on the boundary of G if the function v which is equal to u in the exterior of G and equal to f on Γ is continuous wherever it is defined.

Example. Suppose that the steady-state (independent of time) temperature at every point (x, y, z) outside and on the boundary of a certain body is u. We know that in this case u satisfies the Laplace equation outside G. Thus, to find the steady-state temperature outside G we must solve the exterior Dirichlet problem.

If no restrictions are imposed on the behavior of the solution of the exterior Dirichlet problem at distant points of space, then this problem has many solutions. To insure uniqueness of the solution in the two-dimensional case we require the solution to be bounded. For higher dimensions it suffices to require the solution to go rapidly to zero if at least one of the independent variables goes to infinity. For instance, in the three-dimensional case uniqueness of the solution is secured by requiring that

$$|u(P)| \leqslant M/\overline{AP}. \tag{1, 32}$$

Here M is a constant, A an arbitrary but fixed point, and $\overline{AP}$ denotes the distance between A and P.

For bounded regions the exterior Dirichlet problem can be reduced to the Dirichlet problem which we considered in § 31, and which we now call the interior Dirichlet problem in distinction to the exterior Dirichlet problem. At the same time this reduction clarifies the role of the supplementary conditions at infinity imposed on the solution of the exterior Dirichlet problem (more precisely, on the values of the solution at distant points).

We choose in the region G a point O and a spherical surface (in the two-dimensional case, the circumference of a circle) S of radius R and center O. We next invert the space with respect to S, i.e. we map a point P of the space on the point P^* on the half line OP for which $\overline{OP}\,\overline{OP^*} = R^2$. Under this mapping S remains fixed, and the interior (exterior) of S is mapped on its exterior (interior).

Thus all points of space outside G are mapped on to points of a bounded region G^* surrounding the point O. To every point of G^* other than O there corresponds a unique point outside G. The only point to which there corresponds no point of space under our mapping is the point O. All further considerations must be conducted separately for (*a*) the case of two dimensions (the plane) and (*b*) the case of three dimensions.

We first consider *the case of the plane*. Let u denote a solution of the exterior Dirichlet problem for the region G. We put

$$u^*(P^*) = u(P) \quad \text{and} \quad f^*(P^*) = f(P).$$

The function u^* will be defined at all points of G^* other than O, and will take on the values $f^*(P^*)$ on the boundary of G^*. Using simple calculations one can show that the function $u^*(P^*)$ is a harmonic function of the coordinates of the point P^* (or, briefly, a harmonic function of P^*) provided that $u(P)$ is a harmonic function of P.†

If the function $u(P)$ is bounded, then so is $u^*(P^*)$. Using the theorem on removable singularities, it is possible to define u^* at O in such a manner that the resulting function is harmonic in all of the interior of G^*. It follows, in view of the uniqueness of the solution of the interior Dirichlet problem, that the bounded function u^* is uniquely determined in G^* by its values on the boundary of G^*. This implies the existence and uniqueness of the solution of the exterior Dirichlet problem in the class of bounded functions.

The three-dimensional case. Let u again denote a solution of the exterior Dirichlet problem for the region G. We put

$$u^*(P^*) = (R/\overline{OP^*})\, u(P),$$

or, equivalently, $$u(P) = (R/\overline{OP})\, u^*(P^*).$$

We also put analogously

$$f^*(P^*) = (R/\overline{OP^*}) f(P),$$

or, equivalently, $$f(P) = (R/\overline{OP}) f(P^*).$$

The function u^* is thus defined at all interior points of G^* other than O. On the boundary of G^*, u^* takes on the values f^*. Going over to polar coordinates we can show by means of simple calculations that, if $u(P)$ is a harmonic function of P, then $u^*(P^*)$ is a harmonic function of P^*. If $u(P)$ satisfies the condition (1, 32), then, as is easily seen, $u^*(P^*)$ is bounded. Using the theorem on removable singularities, it is possible to define u^* at O so that the resulting function is harmonic at all interior points of G^*. It follows, in view of the uniqueness of the solution of the interior Dirichlet problem, that the bounded function u^* is uniquely determined in G^* by its values on the boundary of G^*. This implies the uniqueness of the solution of the exterior Dirichlet problem in the class of functions satisfying condition (1, 32).

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† To check this it is necessary to write the Laplace equation in polar coordinates with O as origin. In such coordinate system our transformation is expressed by means of very simple formulas.

It follows from the preceding considerations that if all boundary points of G^* are regular, then the exterior Dirichlet problem for the region G has a solution for every continuous function prescribed on its boundary and this solution satisfies condition (1, 32).

Examples. Let us consider the exterior Dirichlet problem in the case of the plane. Let the prescribed boundary function be identically equal to a constant C. Then C is also a solution of our problem. Moreover, C is the only solution of our problem in the class of bounded functions.

We now consider the exterior Dirichlet problem in three dimensions for the region bounded by the surface of a sphere of radius R and center O. If, as before, the boundary function is identically equal to a constant C, then one solution of our problem is the function

$$u(P) = CR/\overline{OP}, \tag{2,32}$$

and this solution is unique in the class of functions satisfying condition (1, 32).

It is possible to show that the solutions of the two problems below approach the solutions of the above two examples, respectively.

1. Let the temperature on the surface of an infinitely long cylindrical pipe be equal to C. Let the initial temperature of the surrounding air be equal to zero. Then the temperature $u(t, x, y, z)$ of the air at the moment t at the point (x, y, z) tends to C as $t \to \infty$. In physical terms this means that, using an infinitely long pipe of constant temperature C, it is possible to bring the temperature of the surrounding air up to C.

2. Let the temperature of the surface of a sphere of radius R and center O be constant and equal to C. Let the temperature of the surrounding air be everywhere equal to zero. Then the temperature

$u(t,x,y,z)$ of the air at the moment t at the point (x,y,z) approaches the function (2, 32) as $t \to \infty$.

Problem. Using inversion show: (a) the uniqueness of the solution of the exterior Dirichlet problem for an unbounded region G in the plane in the class of bounded functions and (b) the uniqueness of the solution of the exterior Dirichlet problem for an unbounded region G in space in the class of functions satisfying condition (1, 32).

§ 33. The Neumann problem (the second boundary-value problem)

1. *Statement of the (interior) second boundary value-problem. Let there be given a finite region G bounded by a smooth curve with a continuously turning tangent. Let f be a continuous function prescribed on the boundary of G. We wish to find a function u harmonic in the interior of G whose derivative $\partial u/\partial n$ in the direction of the inner normal is equal to the function f at every boundary point of G.* (This problem is also referred to as the Neumann problem.)

In all of this section we assume that u has bounded continuous derivatives of the second order in G and that u, $\partial u/\partial x$ and $\partial u/\partial y$ can be so defined on the boundary of G that the resulting functions are continuous in $\bar{G}$.

We now show that *for a solution of the interior second boundary-value problem to exist it is necessary that the integral of f over the boundary of the region G vanish.*

Proof. We consider the integral

$$\iint_G \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx\,dy.$$

Since u is harmonic, this integral must vanish. By Gauss's theorem we must have

$$\int_\Gamma \left(\frac{\partial u}{\partial n}\right) ds = 0 \quad \text{or} \quad \int_\Gamma f(s)\,ds = 0 \tag{1, 33}$$

(for, by assumption, $\partial u/\partial n = f$ on Γ).

If the region G is multiply-connected and if its boundary consists of a finite number of closed curves each with a continuously turning tangent, then the integral in the equality (1, 33) must be taken over all these curves, and the positive direction on each of them must be chosen so that the region G is always to the left of the boundary.

The same considerations apply in the three-dimensional case. In this case we also find that the integral over the boundary of G of the values of $\partial u/\partial n$ prescribed on the boundary must vanish.

2. If a function u is a solution of a certain Neumann problem, then every other function which differs from u by a constant is also a solution of that problem.

We prove that the solutions of a certain Neumann problem can differ from one another by a constant only. For this it is sufficient to prove that if the function prescribed on the boundary is $\equiv 0$, then the only solutions of the Neumann problem are constants. In fact, let u be a solution of this problem. Then

$$\iint_G u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) dx\,dy = 0. \tag{2, 33}$$

But
$$\iint_G u\frac{\partial^2 u}{\partial x^2}\,dx\,dy = \int_\Gamma u\frac{\partial u}{\partial x}\,dy - \iint_G \left(\frac{\partial u}{\partial x}\right)^2 dx\,dy, \tag{3, 33}$$

$$\iint_G u\frac{\partial^2 u}{\partial y^2}\,dx\,dy = \int_\Gamma u\frac{\partial u}{\partial y}\,dx - \iint_G \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy. \tag{4, 33}$$

In both cases $\int_\Gamma$ denotes integration over all of the boundary of G, traversed in the positive direction. By (2, 33), (3, 33) and (4, 33), we have

$$\iint_G u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) dx\,dy = \int_\Gamma u\frac{\partial u}{\partial n}\,ds - \iint_G \left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy.$$

If u is harmonic in the interior of G and if $\partial u/\partial n = 0$ on the boundary of G, then, by the previous results,

$$\iint_G \left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy = 0.$$

Since $\partial u/\partial x$ and $\partial u/\partial y$ are continuous, this implies that they must both vanish in all of G. Consequently, u is a constant.

This proof is applicable to a region G in any number of dimensions.

Problem. Prove that, for $\partial u/\partial n = 0$ on the boundary of G, every solution u of the Neumann problem for a finite region G is equal to C, provided that the following conditions hold: (1) the boundary of G has continuously varying curvature; (2) it is possible to define u and its *first* derivatives on the boundary of G so that the resulting functions are continuous in $\bar{G}$.

Hint. In this case $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ may be unbounded, and the integrals

$$\iint_G u\frac{\partial^2 u}{\partial x^2}\,dx\,dy \quad \text{and} \quad \iint_G u\frac{\partial^2 u}{\partial y^2}\,dx\,dy$$

may not exist. Consequently, the considerations given before cannot be applied. We must, therefore, approach the region G by means of a region G_ϵ in G such that the directions of the boundaries of G_ϵ and G at nearby points are almost the same. We must transform the integral
$$\iint_{G_\epsilon} u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) dx\,dy$$
just as we did in the case of the integral (2, 33), and we must note that if the boundary of G_ϵ is sufficiently close to the boundary of G, then the absolute value of the integral
$$\int_{\Gamma_\epsilon} u\frac{\partial u}{\partial n}\,ds$$
is small.

3. If the region G is two-dimensional and simply connected, it is easy to reduce the Neumann problem to the Dirichlet problem. Thus, let there exist a solution u of the Neumann problem such that $u, \partial u/\partial x$ and $\partial u/\partial y$ can be extended continuously to the boundary of G (i.e. these functions can be so defined on the boundary of G that the resulting functions are continuous in $\bar{G}$). We now construct in $\bar{G}$ a function v which satisfies in G the Cauchy-Riemann equations
$$\left.\begin{aligned}\partial u/\partial x &= \partial v/\partial y,\\ \partial u/\partial y &= -(\partial v/\partial x).\end{aligned}\right\} \tag{5, 33}$$
The existence of the function v, whose derivatives are defined by (5, 33), follows from the fact that u is harmonic and therefore
$$\frac{\partial^2 v}{\partial y\,\partial x}-\frac{\partial^2 v}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

These equations define v to within a constant term. It is easy to check that at every point of G the derivative of u in some direction $\vec{l}$ is equal to the derivative of v in the direction obtained by rotating $\vec{l}$ counterclockwise by 90°. It can also be checked that for an arbitrary boundary point of G the derivative of u in the direction normal to the boundary is equal to the derivative of v in the direction of the tangent to the boundary. Hence, fixing the value of v at some point A on the boundary of the region, we find that at every point B on the boundary of G
$$v(B)-v(A)=\int_A^B f(s)\,ds, \tag{6, 33}$$
where ds denotes an element of arc length on the boundary of G. Since the integral of $f(s)$ taken over the boundary of G vanishes,

the equality (6, 33) defines v on the boundary of G as a continuous and single-valued function.

It is easy to see that if u is harmonic then v, defined by equations (5, 33), is also harmonic. Therefore, knowing the values of v on the boundary of G, we can determine v uniquely in the interior of G. Then, using (5, 33), we can determine the values of u in G to within a constant term. It is easily verified that the function u determined in this manner is a solution of the original Neumann problem.

In the case of three dimensions it is not possible to reduce the Neumann problem to the Dirichlet problem. In this case it is possible to solve the Neumann problem using integral equations (cf. §§ 34 and 35).

4. *The exterior second boundary-value problem* can be stated as follows. Let there be given a bounded simply-connected region G with a smooth boundary. We wish to find outside G a harmonic function u such that u and its first derivatives can be extended continuously to the boundary of G and such that on the boundary of G the derivative of u in the direction of the outer normal is equal to a prescribed continuous function f. In addition, there are imposed on the solution of our problem the same conditions at infinity which were imposed on the solution at infinity of the exterior Dirichlet problem.

Just as in the case of the exterior Dirichlet problem, it is possible to reduce the exterior second boundary-value problem to the interior second boundary-value problem by means of inversion. What is essential here is the fact that, in view of the conformal nature of the transformation of inversion, angles are preserved. Consequently, a line normal to the original region goes over into a curve normal to the new region. In the two-dimensional case the boundary function for the resulting interior second boundary-value problem (Neumann problem) is obtained as follows (we are using the notation introduced in connexion with the outer Dirichlet problem):

$$u^*(P^*) = u(P), \quad \overline{OP}\,\overline{OP^*} = R^2,$$

$$f^*(s^*) = \frac{\partial u^*}{\partial n^*} = \frac{\partial u}{\partial n}\frac{\partial n}{\partial n^*} = f(s)\frac{\partial n}{\partial n^*}.$$

Here s and s^* denote corresponding points of the original boundary and of the new boundary, and n and n^* denote normals to the two boundaries. In view of the conformality of the mapping, $\partial n/\partial n^*$ is independent of the direction of n. Thus, to compute $\partial n/\partial n^*$ we may assume that the directions n and n^* go through the center O of the mapping. Then

$$\frac{\partial n}{\partial n^*} = \frac{d(\overline{OP})}{d(\overline{OP^*})} = \frac{-R^2}{(\overline{OP^*})^2}.$$

For the exterior second boundary-value problem to have a solution, it is necessary and sufficient that the corresponding interior second boundary-value problem have a solution. For this it is necessary and sufficient that

$$0 = \int_{L^*} f^*(s^*)\,ds^* = \int_L f(s)\,\frac{dn}{dn^*}\,\frac{ds^*}{ds}\,ds = \int_L f(s)\,ds. \qquad (7,33)$$

Here L^* denotes the map of L under inversion. In view of the conformal nature of this mapping,

$$(dn/dn^*)\,(ds^*/ds) = 1.$$

Problem. Is it possible to compute $\partial u/\partial n^*$ for a three-dimensional region in the same way in which $\partial u/\partial n^*$ was computed for a two-dimensional region?

§ 34. Potential theory

1. In §§ 34 and 35 we describe a method of solving the fundamental boundary-value problems for the Laplace equation (and also for the Poisson equation (cf. § 1)) which is independent of the contents of §§ 30–33. This method utilizes certain special representations of the solution in the form of integrals which occur frequently in mechanics and physics and have thus become known as potentials. To obtain a solution of an equation by means of this method, we make use of certain special solutions having singularities of a definite type at a variable point. We observe that this method is essentially applicable to elliptic equations with variable coefficients. As regards the construction of such 'fundamental solutions' for equations more general than the Laplace equation the reader is referred to papers by E. E. Levi and W. Feller.†

Let a point charge q be located at a point O of the space (x, y, z). Then, according to a well-known law of physics, this charge produces at an arbitrary point $Q \neq O$ an electrostatic field whose intensity $\bar{E}$ is given by

$$\bar{E} = kq(\bar{r}/r^3)$$

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† E. E. Levi, *R.C. Circ. mat. Palermo*, vol. xxiv (1907), pp. 275–317. W. Feller, 'Über die Lösungen der linearen partiellen Differentialgleichungen zweiter Ordnung vom elliptischen Typus, *Math. Ann.*, vol. cii (1929–30).

or, in terms of components, by

$$E_x = kq\frac{x-a}{r^3}, \quad E_y = kq\frac{y-b}{r^3}, \quad E_z = kq\frac{z-c}{r^3}. \tag{1,34}$$

Here a, b, c are the coordinates of O; x, y, z are the coordinates of Q, $\bar{r} = \overrightarrow{OQ}$, $r = \overline{OQ}$; and the value of the coefficient k depends on the system of units used.

The right sides of the equations in (1, 34) taken with opposite sign are equal, respectively, to the partial derivatives with respect to x, y, z of the function

$$u(Q) = kq/r + \text{const.} \tag{2,34}$$

This function is called the *potential* of the given electrostatic field. The constant on the right side of (2, 34) is usually taken to be equal to zero so that $u(Q) \to 0$ for $Q \to \infty$. Also, in mathematical work, k is put equal to 1. We shall thus assume that the point charge q gives rise to the potential

$$u(Q) = q/r = q/[(x-a)^2 + (y-b)^2 + (z-c)^2]^{\frac{1}{2}}. \tag{3,34}$$

In view of the fact that the potentials due to point charges are additive, the potential due to a continuous distribution of charge is given by the limit of a sum, i.e. by an integral. In particular, if the charge is distributed over a surface S and if its surface density is $\omega(A)$ (where A belongs to S), then the potential due to this charge is equal to

$$u(Q) = \iint_S \frac{\omega(A)}{r(A, Q)} dS_A. \tag{4,34}$$

Here $r(A, Q)$ is the distance between A and Q, and A is the variable of integration. This is emphasized by the index of the differential (in (4, 34)). If the charge is distributed over a volume V with volume density $\rho(A)$ (A in V), then the potential is given by

$$u(Q) = \iiint_V \frac{\rho(A)}{r(A, Q)} dV_A. \tag{5,34}$$

The right side of (4, 34) is called a simple layer potential and the right side of (5, 34) a volume potential. In the sequel we give necessary and sufficient assumptions for the existence of these integrals.

Now let two charges q and $-q$ be located on an axis l (fig. 12) and let the distance between them be $h > 0$. Let the two charges approach O so that the direction from $-q$ to q always coincides with the positive direction of the axis. Then the potential at any point other than O is given by the difference of quantities which tend to become equal to each other. Consequently this potential tends to zero. If q varies during the motion so that

$$qh = p = \text{const.},$$

then the limiting potential is equal to

$$u(Q) = \lim q\left(\frac{1}{r'} - \frac{1}{r''}\right) = p\frac{\partial(1/r)}{\partial l} = -\frac{p}{r^2}\frac{\partial r}{\partial l} = p\frac{\cos(\overrightarrow{OQ}, l)}{r^2}. \qquad (6,34)$$

In physics, the limiting distribution of the charges is called a *dipole*, the quantity p is called the *moment of the dipole*, and the axis l the *axis of the dipole*. Using point charges one can realize a dipole only approximately (two large charges a small distance apart). In view of the simplicity of dipole fields it is convenient to use them, on a par with point-charge fields, in the investigation of electrostatic fields.

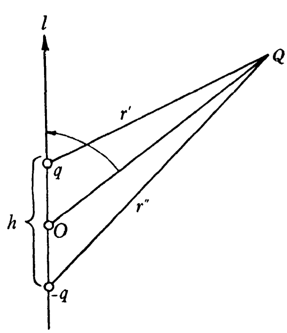

Figure 12

Now, let there be given an oriented surface S, i.e. a surface with indicated exterior and interior. Let there be distributed on S a dipole of moment density $\tau(A)$ (A in S) and let the direction of the dipole axis at A coincide with the direction of the outer normal to S at A. Then the potential of this dipole distribution is equal to

$$u(Q) = \iint_S \frac{\tau(A)\cos(\overrightarrow{AQ}, \bar{n}_A)}{[r(A,Q)]^2}\, dS_A, \qquad (4,34^*)$$

where $\bar{n}_A$ is the outer normal to S at A. The dipole distribution considered can be realized approximately by means of two distributions of charge of density $\tau(A)/h$ and $-\tau(A)/h$, respectively, placed on S a (sufficiently small) distance (measured along the normal to S) $h > 0$ apart. That is why the integral (4, 34*) is called a *double layer potential.*

The right sides of (3, 34) and (6, 34) are harmonic functions at all points of space other than O. This fact can be checked by direct computation. (It suffices to verify the harmonicity of (3, 34) since (6, 34) can then be regarded as the uniform limit of harmonic functions in the neighborhood of every point other than O.) With some minor assumptions on the densities involved, it is now easy to prove that the simple- and double-layer potentials are harmonic everywhere outside S.

Problem 1. Find (a) the simple-layer potential due to a charge distributed uniformly over the surface of a sphere, (b) the volume potential of a charge distributed uniformly over the volume of a sphere.

Problem 2. Let $u = f(r)$, where r is the length of the radius vector in n-dimensional space ($n > 2$) and f is defined for $r > 0$ and has a continuous derivative of the second order. Show that if u is harmonic, then

$$f(r) = c_1 + c_2/r^{n-2}.$$

2. Consider a spatial distribution of charge constant in the z direction. Clearly the electrostatic field due to this distribution of charge will be independent of z. In this case it suffices to study the charge and potential distribution in any of the planes $z = \text{const.}$ Let x and y be the coordinates in one such plane. Here the intensity at a point $Q(x, y)$ is that due to a charge of constant linear density q distributed uniformly along the straight line $x = a$, $y = b$ rather than that due to a point charge. We denote the point (a, b) by the letter O. It follows from considerations of symmetry that for $Q \neq O$ the intensity is equal to

$$\bar{E} = qf(r)\,\bar{r}, \tag{7, 34}$$

where $\bar{r} = \overrightarrow{OQ}$, $r = |\,\bar{r}\,|$. To compute $f(r)$ we take $O = (0, 0)$, $Q = (r, 0)$. Then

$$E_y = E_z = 0, \quad E_x = k\int_{-\infty}^{\infty} \frac{rq}{(r^2+z^2)^{\frac{3}{2}}}\,dz = \frac{kq}{r}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos\phi\,d\phi = \frac{2kq}{r}$$

$$(z = r\tan\phi).$$

Hence, by (7, 34), $\quad f(r) = 2k/r^2,$

so that $\quad \bar{E} = (2kq/r^2)\,\bar{r}.$

Consequently, regardless of the position of the point Q in the (x, y) plane,

$$E_x = \frac{2kq}{r^2}(x-a), \quad E_y = \frac{2kq}{r^2}(y-b).$$

The negatives of these quantities are equal, respectively, to the partial derivatives with respect to x and y of the function

$$u(Q) = 2kq\ln r^{-1} + \text{const.} \tag{8, 34}$$

This function is called the logarithmic potential or, simply, the potential. In mathematical work it is customary to put $2k = 1$ and constant $=0$. Thus in studying a plane field we take the plane potential due to a point charge to be

$$u(Q)=q\ln[r(O,Q)]^{-1}=q\ln[(x-a)^2+(y-b)^2]^{-\frac{1}{2}}. \qquad (9,34)$$

We observe that this potential cannot be obtained from (3, 34) by means of integration along the charged line, for the resulting integral would diverge.

In analogy to para. 1, the potential of a plane dipole is defined by the formula

$$u(Q)=p\frac{\partial \ln(1/r)}{\partial l}=p\frac{\cos(\overrightarrow{OQ},l)}{r}. \qquad (10,34)$$

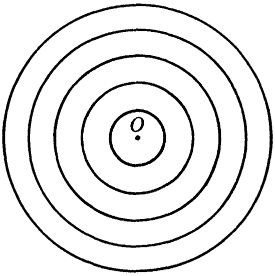

Figure 13

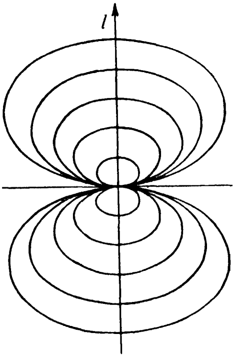

Figure 14

The right sides of (9, 34) and (10, 34) are harmonic in the plane punctured at O (cf. para. 1). The form of the equipotential lines associated with a point charge is represented in fig. 13; that of the equipotential lines associated with a point dipole is represented in fig. 14.

The expressions for the potential of a charge distribution and a dipole distribution are analogous to the expressions (9, 34) and (10, 34) respectively. Here, the place of a volume potential is taken by the two-dimensional potential

$$u(Q)=\iint_G \rho(A)\ln\frac{1}{r(A,Q)}\,dS_A, \qquad (11,34)$$

where G is a plane region. The simple- and double-layer potentials for a plane are, respectively,

$$u(Q)=\int_L \omega(A)\ln\frac{1}{r(A,Q)}\,dl_A, \qquad (12,34)$$

$$u(Q)=\int_L \tau(A)\frac{\cos(\overrightarrow{AQ},\bar{n}_A)}{r(A,Q)}\,dl_A, \qquad (13,34)$$

Here L is a plane curve and $\bar{n}_A$ is a vector whose direction is that of the normal to L at A. We shall regard the curve L as oriented, i.e. having an interior and an exterior. The normal n_A is assumed to be the outer normal.

In the sequel we consider potential theory in the plane only. To develop a theory of potential in spaces of arbitrarily high dimensions one proceeds in a manner analogous to the two-dimensional case.

Problem. Compute the simple-layer potential due to a charge distributed uniformly on the circumference of a circle. (The resulting integral can be computed by means of the theory of residues.)

3. In the sequel we consider a plane curve L with continuously turning tangent. Then, given an arbitrary point P of L, it is possible to choose a coordinate system in which the coordinates (x, y) of P are both zero and the part of L close to P is representable in the form

$$y = \phi(x) \quad (-h \leqslant x \leqslant h;\ h > 0), \tag{14, 34}$$

where $\phi(x)$ is a continuously differentiable function.

Let there be given a function $F(A, Q)$ defined and continuous in A and Q, where A is assumed to be a point of L and Q is assumed to be a point of the plane different from A. Then, if Q varies outside L, the integral

$$w(Q) = \int_L F(A, Q)\, dl_A \tag{15, 34}$$

is always meaningful, and it defines a continuous function of Q. The proof of this fact is elementary.

If $Q = P$ belongs to L, then the integral (15, 34) becomes an improper integral since the integrand is not defined for $A = P$. We shall speak, as usual, of the convergence or divergence of the integral (15, 34) depending on whether the limit

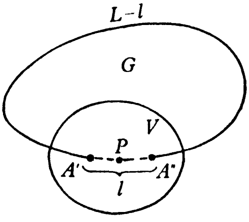

Figure 15

$$\lim_{\substack{A' \to P \\ A'' \to P}} \int_{L-l} F(A, P)\, dl_A, \tag{16, 34}$$

does or does not exist. Here l denotes an arc of the curve L with end-points A' and A'' separated by the point P (cf. fig. 15).

Let l be an arc of L and let P be a point of l which is not an end-point of l. Then P will be referred to as an interior point of l.

We shall say that *the integral* (16, 34) *converges uniformly at a point* P *of* L *if, given an arbitrary* $\epsilon > 0$, *it is possible to find a neighborhood* V *of* P (cf. fig. 15) *and an arc* l *of the curve* L *such that:*

(*a*) P *is an interior point of* l,

(*b*) the integral

$$\int_l F(A, Q)\, dl_A \qquad (17, 34)$$

converges for an arbitrary Q *in* V *and is in absolute value less than* ϵ (the requirement of convergence is essential only if Q is a point of the intersection of l and V).

THEOREM 1. *Let the integral* (16, 34) *converge uniformly at some point* P *of* L. *Then this integral converges at all points* Q *of* L *which are sufficiently close to* P. *Thus formula* (15, 34) *defines the function* $w(Q)$ *in some neighborhood of* P. *This function is continuous at* P.

Proof. We take an arbitrary $\epsilon > 0$ and we choose a neighborhood V and an arc l in accordance with the definition of uniform convergence at a point. Then the integral (17, 34) converges at every point Q interior to l and contained in V. Consequently, the integral (16, 34) also converges at such points Q and the first part of our theorem (concerning the determinacy of $w(Q)$ in some neighborhood of P) is proved.

To verify the continuity of $w(Q)$ at P we assume that Q is a point of V. Then

$$| w(Q) - w(P) | = \left| \int_L F(A, Q)\, dl_A - \int_L F(A, P)\, dl_A \right|$$

$$\leqslant \left| \int_l F(A, Q)\, dl_A \right| + \left| \int_l F(A, P)\, dl_A \right| + \left| \int_{L-l} [F(A, Q) - F(A, P)]\, dl_A \right|$$

$$\leqslant 2\epsilon + \int_{L-l} | F(A, Q) - F(A, P) |\, dl_A.$$

However, if l is fixed, the last integral is $< \epsilon$, provided that Q belongs to a sufficiently small neighborhood of P. This conclusion follows from the uniform continuity of the integrand when A varies in $L - l$ and Q varies in the neighborhood of P mentioned in the preceding sentence. Thus, if Q is sufficiently close to P, then

$$| w(Q) - w(P) | < 3\epsilon,$$

which proves the continuity of $w(Q)$ at P and completes the proof of Theorem 1.

THEOREM 2. *If $\omega(A)$ and $\tau(A)$ are continuous bounded functions, then the simple- and double-layer potentials* (12, 34) *and* (13, 34) *are harmonic everywhere outside L.*

In fact, for Q outside L, the possibility of differentiating the functions (12, 34) and (13, 34) any number of times with respect to the coordinates of the point Q can be proved in exactly the same way in which one proves in mathematical analysis the possibility of differentiating a definite integral with respect to a parameter of which the integrand is a function. Theorem 2 is thus an immediate consequence of the harmonicity of the integrands of (12, 34) and (13, 34).

THEOREM 3. *If Q is a point of L and $\omega(A)$ is continuous and bounded, then the integral* (12, 34) *converges. Thus the simple-layer potential is a function defined and continuous in the whole plane.*

Proof. In view of Theorems 1 and 2 it suffices to verify the uniform convergence of the integral (12, 34) at an arbitrary point P of L. To this end we take the point P as the origin and choose the direction of the coordinate axes so as to be able to write the equation of the portion of L close to P in the form (14, 34). We denote this portion of L by l_h. We then have

$$\left|\int_{l_h}\omega(A)\ln\frac{1}{r(A,Q)}\,dl_A\right| \leqslant \max_L|\omega(A)|\int_{l_h}|\ln r(A,Q)|\,dl_A$$
$$= \max|\omega(A)|\int_{-h}^{h}|\ln[(x-a)^2+(y-b)^2]^{\frac{1}{2}}|\{1+[\phi'(a)]^2\}^{\frac{1}{2}}\,da$$
$$(b=\phi(a)). \quad (18, 34)$$

If V and h are sufficiently small, then the distance between an arbitrary point (x, y) of the region V and an arbitrary point (a, b) of the curve l_h is < 1, so that

$$0\leqslant|x-a|\leqslant[(x-a)^2+(y-b)^2]^{\frac{1}{2}}<1.$$

Using the estimate (18, 34), we now see that for Q in V

$$\left|\int_{l_h}\omega(A)\ln\frac{1}{r(AQ)}\,dl_A\right|$$
$$\leqslant\max_L|\omega(A)|\max_{l_h}\{1+[\phi'(a)]^2\}^{\frac{1}{2}}\int_{-h}^{h}|\ln|x-a||\,da$$
$$\leqslant\max_L|\omega(A)|\max_{l_h}\{1+[\phi'(a)]^2\}^{\frac{1}{2}}\,2\int_0^{2h}|\ln a|\,da\underset{h\to 0}{\to}0$$

uniformly with respect to the point Q varying in V. This proves Theorem 3.

Remark. We proved the convergence of the integral on the left side of (18, 34) (for Q in l_h) while estimating it as in (18, 34), for an improper integral converges as soon as it converges absolutely.

THEOREM 4. *Let L be a closed plane curve with continuously turning tangent bounding a region G. Then the double-layer potential on L with unit density (i.e. the integral* (13, 34) *with $\tau(A) \equiv 1$)*

is equal to -2π if Q is in G,
converges and is equal to $-\pi$ if Q is on L,
is equal to zero if Q is in H, the exterior of $\bar{G}$.

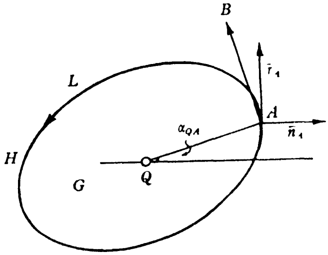

Figure 16

In fact, let Q be an interior point of G and let A traverse L in the positive direction. Let α_{QA} denote the angle between the vector $\overrightarrow{QA}$ and the x-axis. Finally, let $\overrightarrow{AB}$ denote the vector obtained by rotating $\overrightarrow{QA}$ by $+90°$. Then (cf. fig. 16)

$$\frac{d\alpha_{QA}}{dl} = \frac{\cos(\overrightarrow{AB}, \bar{\tau}_A)^\dagger}{r(A,Q)} = \frac{\cos(\overrightarrow{QA}, \bar{n}_A)}{r(A,Q)} = -\frac{\cos(\overrightarrow{AQ}, \bar{n}_A)}{r(A,Q)},$$

hence
$$\int_L \frac{\cos(\overrightarrow{AQ}, \bar{n}_A)}{r(A,Q)}\, dl_A = -\int_L d\alpha_{QA} = -2\pi.$$

The cases Q on L and Q in H are proved analogously.

We now go over to the general case.

THEOREM 5. *Let L be a plane closed curve with continuously turning tangent made up of a finite number of convex arcs.* (*We call an arc convex if every straight line intersects it in at most two points.*) *Let G be the region bounded by L. Some of the arcs which make up L are bent towards and others away from the interior of G.*

Let $\tau(A)$ be bounded and continuous. Then, for Q on L, the integral (13, 34) *converges. Thus the double-layer potential is defined by formula* (13, 34) *in the whole plane. In general, this function has a discontinuity of the first kind on L. More precisely, there exist two*

---

† This is easily checked if we replace differentials by increments and the arc Δl by the tangent to Δl at A.

functions $\underset{\sim}{u}(Q)$ *and* $\tilde{u}(Q)$, *defined in* $G+L$ *and* $H+L$, *respectively, such that*

$$\left.\begin{aligned} u(Q)&=\underset{\sim}{u}(Q) \quad (Q\in G),\\ u(Q)&=\tilde{u}(Q) \quad (Q\in H),\end{aligned}\right\} \tag{19, 34}$$

$$u(Q)=\frac{\tilde{u}(Q)+\underset{\sim}{u}(Q)}{2} \quad (Q\in L),$$

$$\tilde{u}(Q)-\underset{\sim}{u}(Q)=2\pi\tau(Q) \quad (Q\in L).$$

Proof. We take an arbitrary point P in L and we consider together with the potential (13, 34) another double-layer potential

$$u_1(Q)=\int_L \tau(P)\frac{\cos(\overrightarrow{AQ},\bar{n}_A)}{r(A,Q)}\,dl_A=\begin{cases}-2\pi\tau(P) & (Q\in G),\\ -\pi\tau(P) & (Q\in L),\\ 0 & (Q\in H).\end{cases}$$

We form the difference

$$u(Q)-u_1(Q)=\int_L [\tau(A)-\tau(P)]\frac{\cos(\overrightarrow{AQ},\bar{n}_A)}{r(A,Q)}\,dl_A, \tag{20, 34}$$

and prove that the integral on the right converges uniformly at the point $Q=P$. Assuming this to have been proved, we can claim, by Theorem 1, that, at $Q=P$, $u(Q)$ has a discontinuity of the same kind as $u_1(Q)$. This means that the values $u(Q)$ approach a limit as $Q\to P$ in G and, also, as $Q\to P$ in H. The value $u(P)$ exists and is equal to the arithmetic mean of the above two limiting values, and the jump of $u(Q)$ at P, associated with crossing from G to H, is equal to $2\pi\tau(P)$. Considered as a function on G and supplemented by the limiting values obtained by extending it on to the boundary L of G, $u(Q)$ yields the function $\underset{\sim}{u}(Q)$, continuous in $G+L$. An analogous statement holds if G is replaced by H and $\underset{\sim}{u}(Q)$ by $\tilde{u}(Q)$. This suffices to prove Theorem 5.

We must still verify the uniform convergence of the integral (20, 34) at the point P. To this end we choose, as in the proof of Theorem 3, an arc l_h and we estimate an integral of the form (20, 34) taken over l_h. We then obtain

$$\begin{aligned}\left|\int_{l_h}[\tau(A)-\tau(P)]\frac{\cos(\overrightarrow{AQ},\bar{n}_A)}{r(A,Q)}\,dl_A\right| &\leqslant \max_{l_h}|\tau(A)-\tau(P)|\int_{l_h}\left|\frac{\cos(\overrightarrow{AQ},\bar{n}_A)}{r(A,Q)}\right|dl_A\\ &=\max_{l_h}|\tau(A)-\tau(P)|\int_{l_h}|d\alpha_{QA}|\leqslant \max|\tau(A)-\tau(P)|6\pi\xrightarrow[h\to 0]{}0.\end{aligned}$$

We now justify the last step above. In view of the smallness of l_h, we may assume that it consists of no more than two convex arcs. For every convex arc l the following inequality holds:

$$\int_l |d\alpha_{QA}| < 3\pi$$

(cf. fig. 17). For proof note that, if Q is in the unshaded part of the plane or on its boundary, α_{QA} varies monotonically and it increases by less than 2π as A traverses l. If Q is in the part of the plane shaded once, α_{QA} has two sections of monotonicity in each of which it changes by less than π. Finally, if Q is in the part of the plane shaded twice, α_{QA} has three sections of monotonicity in each of which it changes by less than π. This completes the proof of Theorem 5.

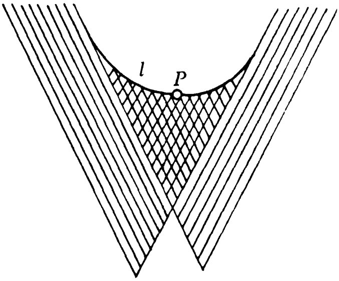

Figure 17

Finally, we consider the normal derivative of a simple-layer potential. Thus, let P be a point of L and let $F(Q)$ be a certain function defined in some neighborhood of P. We introduce the following notation

$$\frac{\partial F(P)}{\partial n^+} = \lim_{P' \to P} \frac{F(P') - F(P)}{r(P', P)},$$

$$\frac{\partial F(P)}{\partial n^-} = \lim_{P'' \to P} \frac{F(P) - F(P'')}{r(P, P'')}.$$

Here n denotes the normal to the curve L at the point P, n^+ its outer portion (with respect to G), and n^- its inner portion. We shall regard as positive the direction $\bar{n}$ of the normal pointing to the exterior of G. The point P' is in H and the point P'' in G.

We assume that L satisfies all the conditions formulated in Theorem 5 and, in addition, has bounded curvature. Then the following theorem holds:

THEOREM 6. *Consider the simple-layer potential* $u(Q)$ *defined by formula* (12, 34). *The derivatives* $\partial u(P)/\partial n^+$ *and* $\partial u(P)/\partial n^-$ *of this* *function exist at an arbitrary point P of L. Also*

$$\frac{\partial u(P)}{\partial n^+} = -\int_L \omega(A)\frac{\cos(\overrightarrow{AP}, \bar{n}_P)}{r(A,P)}\,dl_A - \pi\omega(P), \qquad (21,34)$$

$$\frac{\partial u(P)}{\partial n^-} = -\int_L \omega(A)\frac{\cos(\overrightarrow{AP}, \bar{n}_P)}{r(A,P)}\,dl_A + \pi\omega(P). \qquad (22,34)$$

The integrals on the right sides of (21, 34) *and* (22, 34) *converge. It is assumed that $\omega(A)$ is continuous and bounded.*

Proof. If Q belongs to n_P but not to L, the derivative of $u(Q)$ in the direction n_P exists and its value is obtained by differentiating the integral (12, 34) with respect to the parameter n_P:

$$\frac{\partial u(Q)}{\partial n_P} = -\int_L \omega(A)\frac{\partial \ln r(A,Q)}{\partial n_P}\,dl_A = -\int_L \omega(A)\frac{\cos(\overrightarrow{AQ}, \bar{n}_P)}{r(A,Q)}\,dl_A. \qquad (23,34)$$

We now consider the double-layer potential $u_1(Q)$ due to a dipole distribution on L with density $\omega(A)$. Then, if Q does not belong to L,

$$\frac{\partial u(Q)}{\partial n_P} + u_1(Q) = \int_L \omega(A)\frac{\cos(\overrightarrow{AQ}, \bar{n}_A) - \cos(\overrightarrow{AQ}, \bar{n}_P)}{r(A,Q)}\,dl_A. \qquad (24,34)$$

We prove that the integral obtained converges uniformly at the point P if Q belongs to n_P. Of course, in this case it is necessary to alter the definition of uniform convergence at the point P (cf. para. 3) somewhat; namely, it must be required that Q be a point of the intersection of n_P and a neighborhood V of P rather than an arbitrary point of G. Theorem 1, however, remains valid if in its formulation we require that Q be a point of n_P.

Let l be a small portion of L around P. Then

$$\begin{aligned}
&\left|\int \omega(A)\frac{\cos(\overrightarrow{AQ}, \bar{n}_A) - \cos(\overrightarrow{AQ}, \bar{n}_P)}{r(A,Q)}\,dl_A\right| \\
&\qquad \leqslant C\int_l \frac{|\cos(\overrightarrow{AQ}, \bar{n}_A) - \cos(\overrightarrow{AQ}, \bar{n}_P)|}{r(A,Q)}\,dl_A \\
&\qquad \leqslant 2C\int_l \frac{|\sin\frac{1}{2}\{(\overrightarrow{AQ}, \bar{n}_P) - (\overrightarrow{AQ}, \bar{n}_A)\}|}{r(A,Q)}\,dl_A \\
&\qquad = 2C\int_l \frac{|\sin\frac{1}{2}(\bar{n}_A, \bar{n}_P)|}{r(A,Q)}\,dl_A \leqslant C\int_l \frac{|(\bar{n}_A, \bar{n}_P)|}{r(A,Q)}\,dl_A. \qquad (25,34)
\end{aligned}$$

In view of the assumed boundedness of the curvature $\kappa(A)$ of L,

$$|(\bar{n}_A, \bar{n}_P)| = \left| \int_{\overset{\smile}{AP}} \kappa(A)\, dl_A \right| < C_1 \,|\overset{\smile}{AP}|,$$

and the left side of (24, 34) does not exceed

$$CC_1 \int_l \frac{|\overset{\smile}{AP}|}{r(A,Q)}\, dl_A. \qquad (26, 34)$$

If the arc l is sufficiently small, then, for $A \neq P$,

$$2^{-\frac{1}{2}} < |\sin(\overrightarrow{AP}, \bar{n}_P)| < 1.$$

Denoting by A' the projection of the point A on n_P, we find that

$$r(A,Q) \geqslant r(A,A') > 2^{-\frac{1}{2}} r(A,P) > \tfrac{1}{2}\,|\overset{\smile}{AP}|,$$

and the estimate (25, 34) shows that the left side of (24, 34) is

$$\leqslant CC_1\, 2 \int_l dl = 2CC_1 \,|l| \underset{|l| \to 0}{\longrightarrow} 0,$$

which is what we wished to prove.

The uniform convergence of the integral (24, 34) at the point P implies, by Theorem 1 (altered suitably in view of the fact that Q is a point of the intersection of n_P and V), that this integral is meaningful (converges) for $Q = P$, i.e. it approaches a limit as $Q \to P$ along the straight line n_P, and this limit is equal to the value of the integral (24, 34) at $Q = P$. In other words,

$$\lim_{P' \to P} \left[\frac{\partial u(P')}{\partial n_P} + u_1(P') \right] = \lim_{P'' \to P} \left[\frac{\partial u(P'')}{\partial n_P} + u_1(P'') \right]$$

$$= \int_L \omega(A) \frac{\cos(\overrightarrow{AP}, \bar{n}_A) - \cos(\overrightarrow{AP}, \bar{n}_P)}{r(A,P)}\, dl_A. \qquad (27, 34)$$

However, the nature of the discontinuity of the second term is determined by Theorem 5:

$$\lim_{P' \to P} u_1(P') = \tilde{u}_1(P) = \int_L \omega(A) \frac{\cos(\overrightarrow{AP}, \bar{n}_A)}{r(A,P)}\, dl_A + \pi\omega(P),$$

$$\lim_{P'' \to P} u_1(P'') = \underset{\sim}{u}_1(P) = \int_L \omega(A) \frac{\cos(\overrightarrow{AP}, \bar{n}_A)}{r(A,P)}\, dl_A - \pi\omega(P).$$

This and (27, 34) imply that the limits and the integral

$$\lim_{P' \to P} \frac{\partial u(P')}{\partial n_P}, \quad \lim_{P'' \to P} \frac{\partial u(P'')}{\partial n_P}, \quad \int_L \omega(A) \frac{\cos(\overrightarrow{AP}, \bar{n}_P)}{r(A,P)}\, dl_A$$

exist and

$$\left.\begin{aligned} \lim_{P'\to P}\frac{\partial u(P')}{\partial n_P} &= -\int_L \omega(A)\frac{\cos(\overrightarrow{AP},\vec{n}_P)}{r(A,P)}\,dl_A-\pi\omega(P),\\ \lim_{P'\to P}\frac{\partial u(P'')}{\partial n_P} &= -\int_L \omega(A)\frac{\cos(\overrightarrow{AP},\vec{n}_P)}{r(A,P)}\,dl_A+\pi\omega(P). \end{aligned}\right\} \quad (28,34)$$

Using the mean-value theorem it is not difficult to see that, if some continuous function $f(x)$ is defined in the interval $[a, b]$ $(a < b)$ and if $f(x)$ is differentiable for $a < x < b$ and

$$\lim_{\substack{x\to a\\(x>a)}} f'(x) \qquad (29,34)$$

exists, then $f'(a)$ exists and is equal to (29, 34). For by $f'(a)$ we mean simply

$$\lim_{\substack{\Delta x\to 0\\(\Delta x>0)}}\frac{f(a+\Delta x)-f(a)}{\Delta x}.$$

It therefore follows from the preceding paragraph that $\partial u(P)/\partial n^+$ and $\partial u(P)/\partial n^-$ exist and

$$\frac{\partial u(P)}{\partial n^+}=\lim_{P'\to P}\frac{\partial u(P')}{\partial n_P},\quad \frac{\partial u(P)}{\partial n^-}=\lim_{P''\to P}\frac{\partial u(P'')}{\partial n_P}.$$

This result and (28, 34) prove the validity of (21, 34) and (22, 34). This completes the proof of Theorem 6.

Remark. Theorems 5 *and* 6 *remain valid if the closed curve L which appears in the statements of these theorems is replaced by a* (*not necessarily closed*) *curve* L_0 *with continuously turning tangent, made up of a finite number of convex arcs each having bounded curvature.* To see this we complete L_0 so as to form a closed curve L with continuously turning tangent, made up of a finite number of convex arcs. Let L_1 denote the curve used in the completion process. We prescribe on L_1 a charge or dipole distribution such that the density of the distribution in question is continuous on all of L. Then Theorems 5 and 6 (as applied to L) hold for neighborhoods of all interior points of L_0. Since the integrals (13, 34) and (23, 34) taken over L_1 are continuous near every point P interior to L_0, the correctness of our claim is proved.

Problem. Prove Theorems 3, 4 and 5 for the case when G is a polygon.

Remarks. (1) All theorems on simple- and double-layer potentials proved in this section remain valid if the curve L has bounded curvature.

(2) All theorems proved in this section can be extended in a natural way to simple- and double-layer potentials in three-dimensional space. We need only assume that the curvature of the surface S over which the integrals corresponding to (4, 34) (simple-layer potential) and (4, 34*) (double-layer potential) are taken is everywhere bounded. It turns out that the simple-layer potential is continuous everywhere and that the double-layer potential and the normal derivatives of the simple-layer potential jump near a point Q of the charged surface by $4\pi\tau(Q)$ and $4\pi\omega(Q)$, respectively. (The corresponding values in the plane are $2\pi\tau(Q)$ and $2\pi\omega(Q)$, respectively.) Here $\omega(Q)$ denotes the density of charge distribution and $\tau(Q)$ the density of dipole distribution on the surface S. These densities are supposed continuous. All considerations of the following section can also be extended in a natural way to the case of three dimensions. Proofs of these theorems can be found in O. D. Kellog, *Foundations of Potential Theory*, Springer, Berlin, 1929. For a Russian source see S. L. Sobolev, *Equations of Mathematical Physics*, Gostekhizdat, 1947, pp. 185–97.

§ 35. Application of potential theory to the solution of boundary-value problems

1. *Reduction of boundary-value problems for harmonic functions to integral equations.* Let L be a plane curve with continuously turning tangent and with continuous and bounded curvature. To simplify matters we also assume that L is convex and that its curvature is everywhere positive.

Let $f(P)$ be a continuous function prescribed on L. We shall solve the (interior) Dirichlet problem which, as we know from § 27, consists in finding a function $u(Q)$ continuous in $G+L$ and harmonic in G such that for $P \in L$

$$u(P)=f(P). \tag{1,35}$$

We try to determine this harmonic function in the form of a double-layer potential of a dipole distribution on L of unknown density $\tau(A)$. According to Theorems 2 and 5 of § 34 there corresponds to such a distribution a function $\underset{\sim}{u}(Q)$ continuous in $G+L$

and harmonic in G. By (19, 34) we have, for P in L,

$$\underset{\sim}{u}(P)=\int_L \tau(A)\frac{\cos(\overrightarrow{AP},\bar{n}_A)}{r(A,P)}\,dl_A-\pi\tau(P).$$

It follows that for the boundary condition (1, 35) to be satisfied it is necessary and sufficient that the function $\tau(A)$ satisfy the Fredholm integral equation of the second kind

$$\tau(P)=\frac{1}{\pi}\int_L \tau(A)\frac{\cos(\overrightarrow{AP},\bar{n}_A)}{r(A,P)}\,dl_A-\frac{1}{\pi}f(P). \tag{2, 35}$$

The exterior Dirichlet problem (cf. § 32) is treated analogously. If a solution is sought in the form of a double-layer potential of a dipole distribution of unknown continuous density $\tau(A)$, then, in analogy to (2, 35), we obtain for $\tau(A)$ the equation

$$\tau(P)=-\frac{1}{\pi}\int_L \tau(A)\frac{\cos(\overrightarrow{AP},\bar{n}_A)}{r(A,P)}\,dl_A+\frac{1}{\pi}f(P)\;(P\in L), \tag{3, 35}$$

where $f(P)$ is the prescribed continuous function.

The interior second boundary-value problem can be stated as follows: We wish to find a function $u(Q)$ continuous in $G+L$ and harmonic in G whose normal derivative $\partial u/\partial n^-$ exists at every point of L and is equal to a prescribed continuous function $-f(P)$,

$$\frac{\partial u(P)}{\partial n^-}=-f(P)\quad (P\in L). \tag{4, 35}$$

We seek a solution in the form of a simple-layer potential (12, 34) with an unknown function $\omega(A)$, which we suppose continuous. In view of Theorem 6, § 34, for the boundary condition (4, 35) to be satisfied it is necessary and sufficient that the following relation hold:

$$\omega(P)=\frac{1}{\pi}\int_L \omega(A)\frac{\cos(\overrightarrow{AP},\bar{n}_P)}{r(A,P)}\,dl_A-\frac{1}{\pi}f(P). \tag{5, 35}$$

The exterior second boundary-value problem is stated similarly.. It reduces to the integral equation

$$\omega(P)=-\frac{1}{\pi}\int_L \omega(A)\frac{\cos(\overrightarrow{AP},\bar{n}_P)}{r(A,P)}\,dl_A-\frac{1}{\pi}f(P). \tag{6, 35}$$

Remark. If we tried to solve the interior Dirichlet problem by means of a simple-layer potential with unknown continuous density

$\omega(A)$ of charge distribution, we would be led to an equation

$$\int_L \omega(A) \ln \frac{1}{r(A,P)}\, dl_A = f(P) \quad (P \in L). \tag{*}$$

This is a Fredholm integral equation of the first kind. The theory of such equations is considerably more complex than the theory of equations of the second kind. It can be shown that equation (*) need not have a solution for all continuous $f(P)$. If, for instance, G is a circle or radius 1, the left side of (12, 34) vanishes at its center for an arbitrary function $\omega(A)$.

2. *Investigation of the integral equations obtained.* Let us put

$$K_1(P,A) = \frac{\cos(\overrightarrow{AP}, \bar{n}_A)}{r(A,P)}, \quad K_2(P,A) = -\frac{\cos(\overrightarrow{AP}, \bar{n}_P)}{r(A,P)}$$
$$(A \in L, \quad P \in L, \quad A \neq P).$$

Then
$$K_1(A,P) = K_2(P,A).$$

Thus the kernels of equations (2, 35) and (6, 35) turn out to be the transpose of one another. The same statement holds for the kernels of equations (3, 35) and (5, 35).

If $A \in L$, $P \in L$, $A \neq P$, then the kernel $K_1(P,A)$ is defined and continuous. On the other hand, for an arbitrary point $P_0 \in L$, the kernel approaches a definite limit as $A \to P_0$, $P \to P_0$ $(A \neq P)$. In fact, we note first of all that

$$\lim \frac{\cos(\overrightarrow{AP}, \bar{n}_A)}{r(A,P)} = -\lim \frac{|\tan(\overrightarrow{AP}, T_A)|}{r(A,P_A)}, \tag{7, 35}$$

where T_A is the tangent to L at A, and P_A is the projection of P on T_A. We choose T_A as our x-axis, the point A as the origin, and we direct the y-axis towards the interior of G. We can then write the equation of the portion of L close to A in the form $y = \phi(x)$. Let $\bar{x}$ denote the abscissa of the point P in the above coordinate system. Then, using Taylor's theorem, we get

$$\frac{|\tan(\overrightarrow{AP}, T_A)|}{r(A,P_A)} = \frac{\phi(\bar{x})}{\bar{x}^2} = \tfrac{1}{2}\phi''(\theta\bar{x}) = \tfrac{1}{2}\,|\kappa(M)|\,(1 + [\phi'(\theta\bar{x})]^2)^{\frac{3}{2}}, \tag{8, 35}$$

where the point M (with abscissa $\theta\bar{x}$) lies on L between A and P, and $\kappa(M)$ denotes the curvature at the point M. It follows from (7, 35) and (8, 35) that for $A \to P_0$, $P \to P_0$ $(A \neq P)$

$$\frac{\cos(\overrightarrow{AP}, \bar{n}_A)}{r(A,P)} \to -\tfrac{1}{2}\phi''(P_0) = -\tfrac{1}{2}\kappa(P_0),$$

i.e. it approaches a finite limit, which is what we wished to prove.

If we define $K_1(P,A)$ for $A = P$ by putting

$$K_1(A,A) = -\tfrac{1}{2}\,|\,\kappa(P_0)\,|,$$

the resulting function, which we shall also denote by $K_1(P,A)$, will be continuous in all variables for arbitrary $A \in L$, $P \in L$, and so will be uniformly continuous. The same is true for $K_2(P,A)$.

We shall make use of the theory of integral equations with continuous kernel of the form

$$y(P) = \lambda \int_L K(P,A)\, y(A)\, dl_A + f(P),$$

an exposition of which can be found, for instance, in the author's course on integral equations.† It is true that there integration is carried out over a region in n-dimensional space, whereas we integrate over a plane curve L. However, our integral can be reduced to an ordinary definite integral with respect to a variable s by introducing s as a parameter on L.

We now turn to equation (2, 35). We prove, in accordance with Fredholm's theorem, that this equation has a unique solution, by showing that the corresponding homogeneous equation

$$\tau(P) = \frac{1}{\pi}\int_L \tau(A)\,\frac{\cos(\overrightarrow{AP}, \bar{n}_A)}{r(A,P)}\, dl_A \tag{9, 35}$$

has only the trivial solution.

Let us assume that equation (9, 35) has a non-trivial solution the maximum of whose absolute value is taken on at the point P_0. Then, putting $P = P_0$ in (9, 35), we get

$$\tau(P_0) = \frac{1}{\pi}\int_L \tau(A)\,\frac{\cos(\overrightarrow{AP_0}, \bar{n}_A)}{r(A,P_0)}\, dl_A.$$

Consequently, in accordance with § 34 (cf. the proof of Theorem 4),

$$\int_L [\tau(P_0) + \tau(A)]\,\frac{d\alpha_{P_0A}}{dl_A}\, dl_A = 0. \tag{10, 35}$$

Since $d\alpha_{P_0A}/dl_A > 0$ (in view of the assumed positiveness of the curvature),

$$\tau(P_0) + \tau(A) \equiv 0 \quad (A \in L) \qquad \text{or} \qquad \tau(A) \equiv -\tau(P_0).$$

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† I. G. Petrovsky, *Lectures on Integral Equations*, Gostekhizdat (1948). Also, cf. Courant-Hilbert, *Methods of Mathematical Physics*, vol. I (1953), Interscience, New York–London.

For $\tau(P_0) \neq 0$ this leads to a contradiction at the point $A = P_0$, however. Consequently, $\tau(A) \equiv 0$. Taking into account the fact that equation (6, 35) is the transpose of equation (2, 35), we obtain, by Fredholm's second theorem,

THEOREM 1. *Equation* (2, 35) *of the interior Dirichlet problem and equation* (6, 35) *of the exterior Dirichlet problem each have one and only one solution for an arbitrary continuous function* $f(P)$.

Applying similar reasoning to the homogeneous equation corresponding to (3, 35), we obtain, in place of relation (10, 35), the relation

$$\int_L [\tau(P_0) - \tau(A)] \frac{d\alpha_{P_0A}}{dl_A} dl_A = 0.$$

This implies that $\quad \tau(P_0) - \tau(A) \equiv 0 \quad (A \in L).$

Consequently, the only non-trivial solution of equation (3, 35) is $\tau(A) \equiv \tau(P_0) = \text{const}$. By Fredholm's second theorem, the homogeneous equation corresponding to the transpose (5, 35) of equation (3, 35) also has one (to within a non-zero multiplication factor) solution which we denote by $\bar{\omega}(P)$. Recalling Fredholm's third theorem, we have

THEOREM 2. *Equation* (3, 35) *of the exterior Dirichlet problem has a solution if and only if*

$$\int_L f(A)\bar{\omega}(A)\, dl_A = 0. \tag{11, 35}$$

If this condition holds, the solution of equation (3, 35) is determined to within an arbitrary constant term. Equation (5, 35) of the interior second boundary-value problem has a solution if and only if

$$\int_L f(A)\, dl_A = 0.$$

If this condition holds, the solution is determined to within the term $C\bar{\omega}(P)$, where C is an arbitrary constant.

Problem. Extend the results of paras. 1 and 2 to the case when L is made up of straight-line segments.

3. *Solution of the boundary-value problems.* Theorems 1 and 2 of this section yield immediately the results of §§ 31–33 concerning solvability conditions of the fundamental boundary-value problems. First, it follows from Theorem 1 that for a region G satisfying the restrictions imposed by us there always exists a unique solution of the interior Dirichlet problem representable in the form of a double-layer potential. In view of the proof of uniqueness of

the solution of Dirichlet's problem given earlier, we can say that the integral equation (2, 35) and the interior Dirichlet problem are equivalent.

It also follows from Theorem 2 that a solution of the interior second boundary-value problem exists for functions $f(A)$ prescribed on the boundary for which

$$\int_L f(A)\, dl_A = 0.$$

This is in complete agreement with the results of § 33. But we proved in that section that in this case the solution of the second boundary-value problem is determined to within a constant term. Now the density of the single-layer potential which yields the solution of the interior second boundary-value problem is determined to within a fixed term $C\overline{\omega}(A)$.

It follows that

$$\int_L \overline{\omega}(A) \ln \frac{1}{r(A,Q)}\, dl_A \equiv \text{const.} \quad (Q \in G+L).$$

The function $\overline{\omega}(P)$ has simple physical meaning; namely, it is equal to the density of charge distribution on L when $G+L$ is a conductor. The problem of finding this function is the so-called problem of Robin. It can be shown that at all points of L

$$\overline{\omega}(P) \neq 0. \tag{12,35}$$

In view of its continuity, $\omega(P)$ must therefore be of constant sign.

Going over to the solution of the exterior Dirichlet problem we note that given some function on the boundary it is not always possible to find a dipole distribution which solves this problem (cf. Theorem 2). This is explained by the fact that, as is easily seen, every double-layer potential (13, 34) goes to zero at infinity. On the other hand, in proving the existence and uniqueness of the solution of the exterior Dirichlet problem in § 32 we supposed the solution to be merely bounded at infinity. If the function prescribed on the boundary satisfies condition (11, 35), then there exists a solution of the exterior Dirichlet problem in the form of a double-layer potential. For an arbitrary continuous function $f(P)$ we can proceed as follows. We form the function

$$f_1(P) = f(P) + C^*,$$

where the constant C^* is to be chosen so that $f_1(P)$ satisfies condition (11, 35). This means that we must put

$$C^* = -\int_L f(A)\,\overline{\omega}(A)\,dl_A \Big/ \int_L \overline{\omega}(A)\,dl_A, \tag{13, 35}$$

which can be done since, by (12, 35),

$$\int_L \overline{\omega}(A)\,dl_A \neq 0.$$

We now solve equation (3, 35) with f replaced by f_1. Let one solution be $\tau_1(P)$. Then the function

$$u(Q) = \int_L \tau_1(A)\,\frac{\cos(\overrightarrow{AQ}, \overline{n}_A)}{r(A,Q)}\,dl_A - C^*$$

is the solution of our exterior Dirichlet problem. As regards the constant term, which appears in the expression for the dipole density as determined from equation (3, 35), it will not appear in the solution of the exterior Dirichlet problem, for the potential of a constant dipole distribution vanishes outside G (cf. Theorem 4, § 34).

Finally, we consider the exterior second boundary-value problem. We know that for functions $f(P)$ satisfying the condition $\int_L f(A)\,dl_A = 0$ this problem has a solution which is unique to within a constant term. That Theorem 1 yields the density of charge distribution which represents a solution of the second boundary-value problem for an arbitrary function $f(P)$ is explained by the easily verified fact that, in general, a simple-layer potential is not bounded at infinity. To clarify this boundedness condition we choose in the plane an arbitrary point O. Then

$$\begin{aligned}\int_L \omega(A)\ln\frac{1}{r(A,Q)}\,dl_A &= \int_L \omega(A)\ln\frac{1}{r(O,Q)}\,dl_A + \int_L \omega(A)\ln\frac{r(O,Q)}{r(A,Q)}\,dl_A \\ &= \ln\frac{1}{r(O,Q)}\int_L \omega(A)\,dl_A + \int_L \omega(A)\ln\frac{r(O,Q)}{r(A,Q)}\,dl_A. \end{aligned} \tag{14, 35}$$

In the last sum the second term goes to zero as $Q \to \infty$. Therefore, the necessary and sufficient condition for the simple-layer potential to be regular at infinity is

$$\int_L \omega(A)\,dl_A = 0.$$

At the same time, changing the order of integration and applying Theorem 4, § 34, we get from equation (6, 35)

$$\int_L f(P)\,dl_P = -\pi\int_L \omega(P)\,dl_P - \int_L\left[\int_L \omega(A)\frac{\cos(\overrightarrow{AP},\bar{n}_P)}{r(A,P)}\,dl_A\right]dl_P$$

$$= -\pi\int_L \omega(P)\,dl_P + \int_L \omega(A)\left[\int_L \frac{\cos(\overrightarrow{PA},\bar{n}_P)}{r(P,A)}\,dl_P\right]dl_A$$

$$= -2\pi\int_L \omega(P)\,dl_P.$$

It follows that the condition

$$\int_L f(P)\,dl_P = 0$$

is both necessary and sufficient for the simple-layer potential constructed by means of equation (6, 35) to be bounded at infinity. The relation (14, 35) shows that if this condition holds, the constructed potential goes to zero at infinity. To obtain the general solution of the exterior second boundary-value problem one must add to this potential an arbitrary constant term.

4. *Solution of the boundary-value problems for the circle.* If G is a circle, the solution of the integral equations (2, 35), (3, 35), (5, 35) and (6, 35) is especially simple. In fact, if R denotes the radius of the circle, it is easily checked that for $A \in L$ and $P \in L$,

$$\cos(\overrightarrow{AP},\bar{n}_A) = -\cos(\overrightarrow{AP},\bar{n}_P) = -\frac{1}{2}\frac{r(A,P)}{R},$$

equations (2, 35) and (3, 35) go over into

$$\tau(P) = \mp\frac{1}{2\pi R}\int_L \tau(A)\,dl_A \mp \frac{1}{\pi}f(P) \quad (P \in L), \qquad (15, 35)_{1,2}$$

and equations (5, 35) and (6, 35) go over into

$$\omega(P) = \pm\frac{1}{2\pi R}\int_L \omega(A)\,dl_A - \frac{1}{\pi}f(P). \qquad (16, 35)_{1,2}$$

We solve equation $(15, 35)_1$. To this end we put

$$\int_L \tau(A)\,dl_A = C$$

and integrate both sides of $(15, 35)_1$ over L. We then get

$$C = -C - \frac{1}{\pi}\int_L f(P)\,dl_P, \quad C = -\frac{1}{2\pi}\int_L f(P)\,dl_P.$$

Substituting the value of C in $(15, 35)_1$, we obtain

$$\tau(P) = \frac{1}{4\pi^2 R}\int_L f(A)\,dl_A - \frac{1}{\pi} f(P) \quad (P \in L).$$

It now follows from (13, 34) that for Q in G we have, by Theorem 4, § 34,

$$\begin{aligned} \underset{\sim}{u}(Q) &= \int_L \left[\frac{1}{4\pi^2 R}\int_L f(A)\,dl_A - \frac{1}{\pi} f(A)\right] \frac{\cos(\overrightarrow{AQ}, \bar{n}_A)}{(A, Q)}\,dl_A \\ &= -\frac{1}{2\pi R}\int_L f(A)\,dl_A - \frac{1}{\pi}\int_L f(A)\frac{\cos(\overrightarrow{AQ}, \bar{n}_A)}{r(A, Q)}\,dl_A \\ &= \frac{1}{\pi}\int_L \left[\frac{\cos(\overrightarrow{QA}, \bar{n}_A)}{r(A, Q)} - \frac{1}{2R}\right] f(A)\,dl_A. \end{aligned}$$

We have thus obtained another form of the Poisson integral considered in § 29.

We now go over to equation $(15, 35)_2$. The homogeneous equation corresponding to $(16, 35)_1$ has the non-trivial solution $\bar{\omega}(P) \equiv \text{const.} \neq 0$ (cf. Theorem 2). Hence, in this case, the conditions (11, 35) and $\int_L f(A)\,dl_A = 0$ coincide. If the condition $\int_L f(A)\,dl_A = 0$ holds, equation $(15, 35)_2$ has the solution

$$\tau(P) = (1/\pi) f(P) + C \quad (P \in L),$$

where C is arbitrary. In the general case we get (cf. (13, 35))

$$C^* = -\frac{1}{2\pi R}\int_L f(A)\,dl_A, \quad f_1(P) = f(P) - \frac{1}{2\pi R}\int_L f(A)\,dl_A,$$

$$\tau_1(P) = \frac{1}{\pi} f(P) - \frac{1}{2\pi^2 R}\int_L f(A)\,dl_A + C,$$

$$\begin{aligned} \tilde{u}(Q) &= \int_L \left[\frac{1}{\pi} f(A) - \frac{1}{2\pi^2 R}\int_L f(A)\,dl_A\right] \frac{\cos(\overrightarrow{AQ}, \bar{n}_A)}{r(A, Q)}\,dl_A \\ &\quad + \frac{1}{2\pi R}\int_L f(A)\,dl_A = -\frac{1}{\pi}\int_L \left[\frac{\cos(\overrightarrow{QA}, \bar{n}_A)}{r(A, Q)} - \frac{1}{2R}\right] f(A)\,dl_A. \quad (Q \in H). \end{aligned}$$

Problem. Solve equation $(16, 35)_1$, and in this connexion obtain the solutions of the interior and exterior second boundary-value problems for the circle. In solving the second of these two problems make use of the formula

$$\int_0^\pi \ln(1 + p^2 - 2p\cos\phi)\,d\phi = 0 \quad (-1 < p < 1).$$

5. We consider the Poisson equation

$$\Delta u = f(x,y). \tag{17,35}$$

We assume that the function $f(x,y)=f(P)$, given in a bounded region G, is bounded and has continuous first partial derivatives. We shall try to solve the interior Dirichlet problem for the Poisson equation.† It suffices to find one arbitrary solution of equation (17, 35), continuous in $\bar{G}=G+L$, without taking into consideration the boundary function. In fact, let v be such a solution. Let w be the solution of the interior Dirichlet problem for the Laplace equation under the boundary condition

$$w\Big|_L = u\Big|_L - v\Big|_L .$$

Then

$$u = v + w$$

is a solution of the original problem. Thus the problem of existence and uniqueness of the solution of the interior Dirichlet problem for equation (17, 35) can be completely reduced to the same problem for the Laplace equation. We now prove that we can take as the required special solution the function

$$v(P) = -\frac{1}{2\pi}\iint_G f(A)\ln\frac{1}{r(A,P)}\,ds_A \tag{18,35}$$

(the logarithmic potential of a charge of density $-(1/2\pi)f(A)$*).*

We first verify that the integral (18, 35) converges everywhere and represents a continuous function of P in the plane. For this it is sufficient to verify the uniform convergence of the integral (18, 35) at an arbitrary point $P_0 \in \bar{G}$ (cf. § 34). For an arbitrary $\rho > 0$ we denote by $D_\rho(P_0)$ the interior of the circle of radius ρ and center P_0, and by $G_\rho(P_0)$ the intersection of $D_\rho(P_0)$ and G. It suffices to show that for an arbitrary $\epsilon > 0$ it is possible to find a $\rho > 0$ such that for an arbitrary $P \in D_\rho(P_0)$ the integral

$$\iint_{G_\rho(P_0)} f(A)\ln\frac{1}{r(A,P)}\,ds_A$$

converges and its absolute value is $< \epsilon$. To prove this result we denote by M the least upper bound of $|f|$ on G and we introduce polar coordinates with origin at P. Then, for $\rho \leqslant \frac{1}{2}$,

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† I.e. we seek a solution of (17, 35) continuous in $\bar{G}$ which, on the boundary of G, takes on the values of a continuous function prescribed on that boundary.

$$\left|\iint_{G_\rho(P_0)} f(A)\ln\frac{1}{r(A,P)}\,ds_A\right| \leqslant M\iint_{G_\rho(P_0)} |\ln r(A,P)|\,ds_A$$

$$\leqslant M\int_0^{2\pi}\int_0^{2\rho}(-\ln r)\,r\,dr\,d\phi = 4\pi M\rho^2\left(\ln\frac{1}{2\rho}+\frac{1}{2}\right)_{\rho\to 0}\to 0, \qquad (19,35)$$

which is the desired result.

We show that the integral (18, 35) has continuous first partial derivatives. To this end we first differentiate the integral formally with respect to x, without worrying about convergence. With (x, y) denoting the coordinates of P and with (a, b) denoting the coordinates of A, we get

$$\phi(P) = \frac{1}{2\pi}\iint_G f(A)\frac{x-a}{[r(A,P)]^2}\,ds_A. \qquad (20,35)$$

Just as in (19, 35), it is easy to verify the uniform convergence of this integral at every point of $\bar{G}$. It follows that this integral represents a function continuous in the whole plane. To show that $\phi(A) \equiv v'_x(A)$, we choose an arbitrary but fixed point $P(x, y) \in G$ and a point $P_1(x+h, y)$ $(h \neq 0)$.† Then

$$\left|\phi(P) - \frac{v(P_1)-v(P)}{h}\right| = \frac{1}{2\pi}\left|\iint_G f(A)\frac{x-a}{[r(A,P)]^2}\,ds_A\right.$$

$$\left. -\frac{1}{h}\left(-\iint_G f(A)\ln\frac{1}{r(A,P_1)}\,ds_A + \iint_G f(A)\ln\frac{1}{r(A,P)}\,ds_A\right)\right|$$

$$\leqslant \frac{1}{2\pi}\left|\iint_{G_\rho(P)} f(A)\frac{x-a}{[r(A,P)]^2}\,ds_A\right| + \frac{1}{2\pi h}\left|\iint_{G_\rho(P)} f(A)\ln\frac{r(A,P)}{r(A,P_1)}\,ds_A\right|$$

$$+\frac{1}{2\pi}\left|\iint_{G-G_\rho(P)} f(A)\left(\frac{x-a}{[r(A,P)]^2} - \frac{1}{h}\ln\frac{r(A,P_1)}{r(A,P)}\right)ds_A\right|. \qquad (21,35)$$

As a result of the uniform convergence of the integral (20, 35) at the point P, the first of the integrals obtained goes to zero with ρ. The second integral also goes to zero if $0 < |h| < \rho$. To prove this we break up $G_\rho(P)$ into two parts, $G'_\rho(P)$ and $G''_\rho(P)$, where:

$G'_\rho(P)$ consists of points of $G_\rho(P)$ for which $r(A,P) > r(A,P_1)$,

$G''_\rho(P)$ consists of points of $G_\rho(P)$ for which $r(A,P) \leqslant r(A,P_1)$.

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† Proceeding in exactly the same way as in Theorem 2 on p. 203, it is easy to show that the function $v(P)$ has derivatives of all orders and is harmonic.

Taking into account the fact that $\ln(1+\delta) < \delta$ for $\delta > 0$, we get

$$\frac{1}{2\pi h}\left|\iint_{G_\rho(P)} f(A)\ln\frac{r(A,P)}{r(A,P_1)}ds_A\right|$$

$$\leqslant \frac{M}{2\pi h}\left(\iint_{G'_\rho(P)} \ln\frac{r(A,P)}{r(A,P_1)}ds_A + \iint_{G''_\rho(P)} \ln\frac{r(A,P_1)}{r(A,P)}ds_A\right)$$

$$\leqslant \frac{M}{2\pi h}\left(\iint_{G'_\rho(P)} \frac{r(A,P)-r(A,P_1)}{r(A,P_1)}ds_A + \iint_{G''_\rho(P)} \frac{r(A,P_1)-r(A,P)}{r(A,P)}ds_A\right)$$

$$\leqslant \frac{M}{2\pi}\left(\iint_{G'_\rho(P)} \frac{ds_A}{r(A,P_1)} + \iint_{G''_\rho(P)} \frac{ds_A}{r(A,P)}\right)^{\dagger}$$

$$\leqslant \frac{M}{2\pi}2\int_0^{2\pi}\int_0^{2\rho}\frac{1}{r}r\,dr\,d\phi = 4M\rho.$$

$\rho > 0$ can be chosen so small that the first two integrals on the right side of (21, 35) are $< \frac{1}{3}\epsilon$, where $\epsilon > 0$ is an arbitrary prescribed number. Keeping this ρ fixed and decreasing $|h|$, we can make the last integral in (21, 35) smaller than $\frac{1}{3}\epsilon$, for, as $|h| \to 0$, the integrand goes uniformly to zero in $G - G_\rho(P)$.

v'_y is treated in an analogous manner. Thus

$$v'_x(P) = \frac{1}{2\pi}\iint_G f(A)\frac{x-a}{[r(A,P)]^2}ds_A,$$
$$v'_y(P) = \frac{1}{2\pi}\iint_G f(A)\frac{y-b}{[r(A,P)]^2}ds_A.$$

Thus far we have used only the boundedness of the continuous‡ function $f(P)$. In the sequel we make use of the fact that $f(P)$ has continuous first partial derivatives. We fix $P_0 \epsilon G$ and choose ρ so small that $D_\rho(P_0) \epsilon G$. Then the integral

$$v_1(P) = -\frac{1}{2\pi}\iint_{G-D_\rho(P_0)} f(A)\ln\frac{1}{r(A,P)}ds_A$$

has continuous partial derivatives of all orders in $D_\rho(P_0)$ ($\rho > 0$), and satisfies the equation

$$\Delta v_1 = 0. \qquad (22, 35)$$

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† Since $|r(A,P) - r(A,P_1)| \leqslant h$.

‡ The continuity of $f(P)$ is not essential, provided that all integrals involved exist either as Riemann integrals or as Lebesgue integrals.

That this is so follows from the fact that, if P is a point of $D_\rho(P_0)$, it is possible to differentiate this integral under the integral sign any number of times with respect to the coordinates of the point P. It thus suffices to consider the integral

$$v_2(P) = -\frac{1}{2\pi}\iint_{D_\rho(P_0)} f(A)\ln\frac{1}{r(A,P)}\,ds_A.$$

We integrate the expression for $\partial v_2/\partial x$ by parts and get

$$\begin{aligned}\frac{\partial v_2(P)}{\partial x} &= \frac{1}{2\pi}\iint_{D_\rho(P_0)} f(A)\frac{x-a}{[r(A,P)]^2}\,ds_A\\ &= \frac{1}{2\pi}\iint_{D_\rho(P_0)}\left\{-\frac{\partial}{\partial a}[f(A)\ln r(A,P)] + f_a'(A)\ln r(A,P)\right\}da\,db\\ &= -\frac{1}{2\pi}\int_{C_\rho(P_0)} f(A)\ln r(A,P)\,db\\ &\quad -\frac{1}{2\pi}\iint_{D_\rho(P_0)} f_a'(A)\ln\frac{1}{r(A,P)}\,ds_A. \qquad (23,\ 35)\end{aligned}$$

Here $C_\rho(P_0)$ denotes the circumference of the circle $D_\rho(P_0)$, and the direction of integration is that of the positive direction on $C_\rho(P_0)$. We saw earlier (cf. (19, 35)) that the last integral (in (23, 35)) has continuous first partial derivatives which, ρ being sufficiently small, are arbitrarily small for $P \in D_\rho(P_0)$. As for the first integral on the right side of (23, 35), it can be differentiated in $D_\rho(P_0)$ with respect to x and y without any restrictions, for the point P does not lie on the curve of integration. We thus have proved the existence and continuity in $D_\rho(P_0)$ of the second partial derivatives of $v_2(P)$. At the same time we have proved the existence and continuity in G of the second partial derivatives of $v(P)$.

Further, for $P \in D_\rho(P_0)$,

$$\frac{\partial^2 v_2}{\partial x^2} = -\frac{1}{2\pi}\int_{C_\rho(P_0)} f(A)\frac{x-a}{[r(A,P)]^2}\,db + \eta_1(P,\rho),$$

where $\eta_1(P,\rho)$ goes uniformly to zero with ρ, provided that $P \in D_\rho(P_0)$. Likewise

$$\frac{\partial^2 v_2}{\partial y^2} = \frac{1}{2\pi}\int_{C_\rho(P_0)} f(A)\frac{y-b}{[r(A,P)]^2}\,da + \eta_2(P,\rho).$$

Going over to polar coordinates with origin at $P_0(x_0, y_0)$ and making use of (22, 35), we find that

$$\begin{aligned}\Delta v &= \frac{\partial^2 v_2(P_0)}{\partial x^2} + \frac{\partial^2 v_2(P_0)}{\partial y^2} \\ &= \frac{1}{2\pi}\int_{C_\rho(P_0)} \frac{f(A)}{[r(A,P)]^2}[(y_0-b)\,da-(x_0-a)\,db] \\ &\qquad + \eta_1(P_0,\rho)+\eta_2(P_0,\rho) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x_0+\rho\cos\phi, y_0+\rho\sin\phi)(\sin^2\phi+\cos^2\phi)\,d\phi \\ &\qquad + \eta_1+\eta_2 \underset{\rho\to 0}{\to} f(P_0)\end{aligned}$$

which is what we set out to prove.

We observe that it is possible to weaken the conditions imposed on the right side of equation (17, 35). However, it is not enough to require that $f(P)$ be continuous and bounded, for then the integral (18, 35) may not possess partial derivatives of the second order. That is why I. I. Privalov constructed (in 1925) the generalized Laplace operator

$$\Delta^*\phi(P) = \lim_{\rho\to 0}\frac{4}{\rho^2}\left[\frac{1}{\pi\rho^2}\iint_{D_\rho(P)}\phi(A)\,ds_A - \phi(P)\right].$$

It turns out that, if $\phi(P)$ has in G continuous partial derivatives of the second order, then, for $P \in G$, $\Delta^*\phi(P)$ exists and is identically equal to $\Delta\phi(P)$. On the other hand, if $f(P)$ is continuous and bounded in the bounded region G and $v(P)$ is represented by means of formula (18, 35), $\Delta^* v(P)$ exists and

$$\Delta^* v(P) \equiv f(P).$$

Remark. All considerations of this section can be carried over in a natural way to the case of the Newtonian potential (5, 34) of a charged region in three-dimensional space. If we assume that the charge density $\rho(A)$ is continuous together with its first derivatives, the potential $u(Q)$ and its first derivatives turn out to be continuous everywhere. The potential $u(Q)$ is harmonic outside the charged domain and satisfies the Poisson equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -4\pi\rho$$

in its interior.

§36. Approximate solution of the Dirichlet problem by the method of finite differences

Let f be a continuous function prescribed on the boundary of a finite region G. We assume that there exists a harmonic function u which takes the prescribed values f on the boundary of G. In 1925, L. A. Lyusternik† gave a method for the approximate computation of u. While this method applies to spaces of any number of dimensions we restrict our presentation, for reasons of simplicity, to the case of a two-dimensional region. In our presentation certain statements are at first left unproved. These statements appear in italics and their proofs are given somewhat later.

Let our region G be a region in the (x, y) plane. We draw in this plane two families (a mesh) of straight lines parallel to the coordinate axes

$$x = mh \quad \text{and} \quad y = nh.$$

Here h is a certain positive number, and m and n take on successive integral values such that the whole region G is covered by squares of side length h. We call the vertices of these squares lattice or net points of the mesh. It is our aim to determine approximately the values of u at the net points in the interior of G. These approximate values are denoted by u_h.

We choose an arbitrary $\epsilon > 0$ and denote by Γ_ϵ the totality of squares, each of which has the property that the distance of one of its vertices from the boundary of G is at most equal to ϵ. The value assigned to u_h at a vertex of a square of Γ_ϵ is the value of f at a boundary point of G closest to that vertex. For sufficiently small h and ϵ, the values of u_h defined as above at the net points of Γ_ϵ differ by as little as we please from the values of u at these points. In fact, the function which coincides with u in G and with f on the boundary of G is uniformly continuous in $\bar{G}$. Consequently, its values at two points P_1 and P_2 of $\bar{G}$ differ from each other by an arbitrarily small amount if only the distance $\overline{P_1P_2}$ is sufficiently small. In the sequel we always suppose $h < \epsilon$.

The points of the region G belonging to those closed squares which do not belong to Γ_ϵ form one or more polygons M. Since the net points on the boundary of each of these polygons belong to Γ_ϵ, the values of u_h are defined at these points. As for the values of u_h at net points in the interior of these polygons, we define them as the solution of a certain system of linear equations whose number is equal to the number of the undefined values of u_h, i.e. to the number of net points in the interior of G which do not belong to Γ_ϵ. This system of equations is constructed as follows. For an interior net point (x,y) we write the equation

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† Lyusternik himself did not assume the existence of the solution of the Dirichlet problem. He used the method of finite differences to prove the existence of the solution of this problem under certain assumptions about the boundary of G. However, his proof did not extend directly to regions of dimensions higher than two.

$$u_h(x,y) = \tfrac{1}{4}[u_h(x+h,y) + u_h(x-h,y) + u_h(x,y+h) + u_h(x,y-h)]$$

or

$$u_h(x+h,y) + u_h(x-h,y) + u_h(x,y+h) + u_h(x,y-h) - 4u_h(x,y) = 0. \tag{1, 36}$$

If any of the points $(x+h,y)$, $(x-h,y)$, $(x,y+h)$, $(x,y-h)$ belongs to Γ_ϵ, then in (1, 36) the corresponding u_h is replaced by the previous value of u_h at this point. It can be shown that *the system of equations* (1, 36) *always has a unique solution* (Theorem 1), and that *choosing first a sufficiently small* ϵ *and then decreasing sufficiently the value of* h *we arrive at values* u_h *which differ by as little as we please from the corresponding values of* u (Theorem 2); *the smallness of* h *which is necessary for this result to hold depends on* ϵ. If h is small and, consequently, the number of equations large, it is difficult to solve the system of equations (1, 36) by means of the ordinary rules of algebra. However, *to obtain an approximate solution of the system* (1, 36) *one can use in a simple way the method of successive approximations* (Theorem 3).

Equation (1, 36) is the analogue in finite differences of the differential equation of Laplace. In fact, let us assume that the function u has bounded derivatives of order up to four in G. Let us also assume that the points $(x+h,y)$, $(x-h,y)$, $(x,y+h)$, $(x,y-h)$ as well as the segments between the point (x,y) and the points $(x+h,y)$, $(x-h,y)$, $(x,y+h)$, $(x,y-h)$, respectively, lie in G. Then

$$\begin{aligned}
u(x+h,y) &= u(x,y) + hu'_x(x,y) \\
&\quad + \tfrac{1}{2}h^2u''_{xx}(x,y) + \tfrac{1}{6}h^3u'''_{xxx}(x,y) + \tfrac{1}{24}h^4u''''_{xxxx}(\tilde{x},y), \\
u(x-h,y) &= u(x,y) - hu'_x(x,y) \\
&\quad + \tfrac{1}{2}h^2u''_{xx}(x,y) - \tfrac{1}{6}h^3u'''_{xxx}(x,y) + \tfrac{1}{24}h^4u''''_{xxxx}(\tilde{\tilde{x}},y), \\
u(x,y+h) &= u(x,y) + hu'_y(x,y) \\
&\quad + \tfrac{1}{2}h^2u''_{yy}(x,y) + \tfrac{1}{6}h^3u'''_{yyy}(x,y) + \tfrac{1}{24}h^4u''''_{yyyy}(x,\tilde{y}), \\
u(x,y-h) &= u(x,y) - hu'_y(x,y) \\
&\quad + \tfrac{1}{2}h^2u''_{yy}(x,y) - \tfrac{1}{6}h^3u'''_{yyy}(x,y) + \tfrac{1}{24}h^4u''''_{yyyy}(x,\tilde{\tilde{y}}).
\end{aligned}$$

Here $\tilde{x}$ and $\tilde{\tilde{x}}$ denote numbers contained between x and $x+h$, x and $x-h$, and $\tilde{y}$ and $\tilde{\tilde{y}}$ denote numbers contained between y and $y+h$, y and $y-h$. Clearly,

$$\begin{aligned} u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y) \\ = h^2[u''_{xx}(x,y)+u''_{yy}(x,y)]+\tfrac{1}{6}h^4M_1\theta \quad (-1\leqslant\theta\leqslant 1), \end{aligned}$$

where M_1 denotes the least upper bound of the values of $|u''''_{xxxx}|$ and of $|u''''_{yyyy}|$. Therefore, the left side of the equation

$$\frac{u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)}{h^2}=0$$

is equivalent to the expression $u''_{xx}+u''_{yy}$ to within terms of order h^2.

THEOREM 1. *The system of equations* (1, 36) *has always a unique solution.*

Proof. We rewrite this system so that there remain on the left sides of the equations the values of u_h at interior net points of the polygon M and on the right sides, the values of u_h at those net points which are boundary points of the polygon M, i.e. at the points of Γ_ϵ. We recall that the latter values have been defined so that we may regard the right sides of our (rewritten) equations as known. Let our system now be of the form

$$\sum_{j=1}^{N} a_{ij}u_j=f_i \quad (i=1,2,\dots,N), \tag{2, 36}$$

where N is the number of interior net points of the polygons M. We have numbered all the interior net points of the polygons and denoted by u_j the value of u_h at the jth (net) point. The right sides of (2, 36) are linear combinations of the values of u_h at the boundary net points of the polygons M.

It is known from algebra that for the system (2, 36) to have a unique solution for arbitrary values of the f_i it suffices that the corresponding homogeneous system have only the trivial solution. Thus, to prove our theorem we assume that the system

$$\sum_{j=1}^{N} a_{ij}u_j=0 \quad (i=1,2,\dots,N) \tag{3, 36}$$

has a non-trivial solution. We denote by B the largest of the numbers $|u_j|$ $(j=1,\dots,N)$ which, by our assumption, is positive. Without loss of generality, we may assume that B is equal to one of the u_j, for the case of $-B$ equal to one of the u_j can be reduced to the preceding case by changing the signs of all the u_j. Thus, let B be equal to u_{j_0}. Since u_{j_0} is equal to the arithmetic mean of the values u_j at four neighboring[†] net points, each of these four values must be equal to B (for, since the values of u_h at points neighboring the jth point cannot exceed B, they cannot be less than B). Applying this reasoning to each of the neighboring net points of the j_0th net point we find that at *their* neighboring net points the u_j are also equal to B. Continuing in this manner we find that $u_j = B$ at all net points belonging to the boundary of some polygon M and neighboring on the same interior point P of M. This contradicts the fact that the right sides f_i of all the equations (3, 36) are equal to zero. In fact, the f_i are linear combinations of the values of u_h at the points of Γ_ϵ, with coefficients equal to -1. (The latter conclusion is easily derived from a comparison of the formulas (2, 36) and (1, 36).) It follows that, if the values of u_h are equal to $B > 0$ at all points of Γ_ϵ neighboring on P, the f_i cannot all vanish.

THEOREM 2. *Let u denote the exact solution of the Dirichlet problem above and u_h the solution of the system* (1, 36) *under the boundary conditions given above. Then, choosing $\epsilon > 0$ sufficiently small and subsequently decreasing the value of h sufficiently, we obtain values u_h which differ by as little as we please from the values of u at corresponding points.*

Proof. We show that for h sufficiently small, $|u_h - u| < \delta$ at all net points of the mesh. To this end we place the origin in the interior of the region G. Next we choose ϵ so small that $\max |u_h - u| < \frac{1}{2}\delta$ at the points of Γ_ϵ and consider an auxiliary function v_h defined as follows:

$$v_h = u_h - u - \frac{\delta}{2D^2}(D^2 - x^2 - y^2) - \tfrac{1}{2}\delta.$$

Here D denotes the diameter of the region G, i.e. the least upper bound of the distances between pairs of points of G.

Since $|u_h - u| < \frac{1}{2}\delta$ and $D^2 > x^2 + y^2$ at the net points of Γ_ϵ, $v_h \leqslant 0$ at these points. We now show that for h sufficiently small, $v_h < 0$ at all net points belonging to M so that $u_h - u < \delta$ there. To this end we apply to v_h the operator Δ_h which associates with a function $\phi(x, y)$ the function

$$\phi(x+h, y) + \phi(x-h, y) + \phi(x, y+h) + \phi(x, y-h) - 4\phi(x, y).$$

[†] By the neighboring net points of (x, y) we mean the points $(x+h, y)$, $(x-h, y)$, $(x, y+h)$ and $(x, y-h)$.

It is clear that

$$\Delta_h v_h = \Delta_h u_h - \Delta_h u + \frac{\delta}{2D^2}\Delta_h(x^2+y^2).$$

But $\Delta_h u_h = 0, \quad \Delta_h(x^2+y^2) = 4h^2, \quad |\Delta_h u| < \frac{1}{6}M_2 h^4,$

where M_2 is the least upper bound of the values of $|u''''_{xxxx}|$ and $|u''''_{yyyy}|$ in the polygon M. Consequently,

$$\Delta_h v_h > 4h^2\frac{\delta}{2D^2} - \frac{M_2 h^4}{6}$$

and for h sufficiently small $\Delta_h v_h > 0$.

It is now easily seen that v_h cannot take on its largest value in the *interior* of a polygon M. If we add to this the fact that v_h is negative on the boundary of the polygons M, it follows that $v_h < 0$ at all interior points of the polygons M.

Analogous considerations applied to the function

$$w_h - u = u_h - \frac{\delta}{2D^2}(D^2 - x^2 - y^2) - \tfrac{1}{2}\delta$$

lead to the conclusion that $u - u_h < \delta$. The two results imply that, for h sufficiently small, $|u - u_h| < \delta$, which is what we wished to prove.

THEOREM 3. Thus far it made no difference to us in what order the net points in the interior of the polygons M were numbered. At this point, however, we find it of importance to number these points in the following manner. The first point must have as one of its neighboring net points a boundary point of one of the polygons M. The second net point must have as one of its neighboring net points a boundary point of one of the polygons M or the first point, etc. With this numbering of the net points it is possible to solve approximately the system of equations (1, 36) as follows. First we prescribe arbitrary values $u_1, u_2, \ldots, u_N$. We denote these by u_k^0 and refer to them as the zero-th approximation to the solution of the system (1, 36). To describe the method of obtaining successive approximations it is convenient to think of the u_k^0 as being written down next to the proper net points. To obtain the first approximation we erase the value u_1^0 at the first net point, and we write instead the value $u_1^{(1)}$ equal to the arithmetic mean of the values of the u_k^0 at the four neighboring net points of the first net point. We next erase the value u_2^0 which appears next to the second net point and replace it by the arithmetic mean $u_2^{(1)}$ of the values which appear next to the four neighboring net points of the second net point (at one of these there may appear the value $u_1^{(1)}$), etc. Treating all net points in this manner we obtain the values $u_k^{(1)}$ $(k=1,\ldots,N)$. The values $u_k^{(2)}$ $(k=1,\ldots,N)$ of the second approximation are obtained from the values $u_k^{(1)}$ in exactly the same manner in which the values $u_k^{(1)}$ were obtained from the values u_k^0. Subsequent approximations are obtained analogously.

We claim that for all k $(k=1,\ldots,N)$

$$u_k^{(n)} \to u_k \quad \text{as} \quad n \to \infty.$$

Here the u_k $(k=1,\ldots,N)$ form the exact solution of the system (1, 36).

Proof. Let us put $\quad u_k^{(n)} - u_k = v_k^{(n)}.$

It is our aim to show that

$$v_k^{(n)} \to 0 \quad \text{as} \quad n \to \infty.$$

For this we note that the numbers $v_k^{(n+1)}$ are obtained from the numbers $v_k^{(n)}$ $(k=1,2,\ldots,N)$ in the same way in which the $u_k^{(n+1)}$ are obtained from the $u_k^{(n)}$, i.e. $v_k^{(n+1)}$ is the arithmetic mean of the values $v_k^{(n)}$ at the four net points neighboring on the kth net point. Here it must be observed that, if one of these neighboring net points happens to be a boundary point of a polygon, then, at this point, $v_k^{(1)}$ is put equal to zero. It follows that

$$|v_1^{(1)}| \leqslant (\tfrac{3}{4})A, \quad A = \max\{|v_1^0|, |v_2^0|, \ldots, |v_N^0|\},$$

since one of the neighboring (net) points of the first net point is a boundary net point. Analogously,

$$|v_2^{(1)}| \leqslant \left(1 - \frac{1}{4^2}\right)A, \quad |v_3^{(1)}| \leqslant \left(1 - \frac{1}{4^3}\right)A, \ldots,$$

$$|v_N^{(1)}| \leqslant \left(1 - \frac{1}{4^N}\right)A = \alpha A, \quad \text{where} \quad \alpha < 1.$$

It is found in the same way that for all n and k,

$$|v_k^{(n)}| < \alpha^n A,$$

so that $\quad v_k^{(n)} \to 0 \quad \text{as} \quad n \to \infty.$

In theory this relation holds for an arbitrary choice of the zero-th approximation. It is easy to guess, however, that to obtain quickly a good approximation to the exact solution of the system

(1, 36) it is practical to choose the zero-th approximation so that it does not differ by very much from the exact solution of the Dirichlet problem. Then the process of successive approximations converges rapidly. It is usually broken off for values of n for which the values $u_k^{(n)}$ cease to change abruptly as n is increased. Such $u_k^{(n)}$ are then taken as the approximate solution of the system (1, 36).

If for some value of h we obtain a number of polygons M, the system (1, 36) breaks up into a number of independent systems each of which corresponds to one polygon. Each of these systems is solved independently.

§ 37. Survey of the most important results for general elliptic equations

1. It was proved in § 31 that given a simply connected region it is possible to solve the Dirichlet problem for the Laplace equation in the case of two independent variables, provided that the function prescribed on the boundary of the given region is continuous. For a simply connected three-dimensional region it is not always possible to solve the Dirichlet problem. In the case of a three-dimensional region all those boundary points are regular which can be touched from the outside by the vertex of a cone K obtained by rotating about the x_1-axis the curve

$$x_2 = f(x_1) = x_1^k,$$

where k is an arbitrary positive number. This condition can be stated more precisely as follows: Let the region G be located in the space (x_1, x_2, x_3). A boundary point P of G is regular if it is possible to choose in (x_1, x_2, x_3) a coordinate system with origin at P such that all interior points of a cone K with positive abscissas x_1 not exceeding a certain positive number η are outside the region G. On the other hand, Lebesgue† and Urysohn‡ showed independently that a boundary point P of G is not regular if there exists a neighborhood U_P of P such that for a suitable choice of the coordinate axes all points of U_P which do not belong to G do not get out of the cone formed by rotating the curve

$$x_2 = \exp\{-1/x_1\}, \quad (x_1 > 0)$$

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† Lebesgue, 'Sur le problème de Dirichlet', *R.C. Circ. mat. Palermo*, vol. XXIV (1907), pp. 371–402.

‡ P. S. Urysohn, 'Zur ersten Randwertaufgabe der Potentialtheorie, Ein Fall der Unlösbarkeit', *Math. Z.* vol. XXIII (1925), pp. 155–8.

about the x_1-axis. The same is true if this curve is replaced by the curve

$$x_2 = F(x_1) = \exp\{-|\ln x_1|^{1+\epsilon}\} = x_1{}^{|\ln x_1|^{\epsilon}},$$

where ϵ is an arbitrary positive number.

In n-dimensional space $(n>3)$ the role of the function $f(x_1)$ is taken by the function

$$\frac{x_1}{|\ln x_1|^{1/(n-3)}}, \tag{1,37}$$

and that of the function $F(x_1)$ is taken by the function

$$\frac{x_1}{|\ln x_1|^{1/(n-3)+\epsilon}} \tag{2,37}$$

where ϵ is an arbitrary positive number. To obtain the equations of the corresponding cones we need only put (1, 37) and (2, 37) equal to $(x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$.

A necessary and sufficient condition for a boundary point P of G to be regular was found by Wiener.†

2. Consider the linear elliptic equation with variable coefficients:

$$\sum_{i,j=1}^{n} a_{ij}(x_1 \dots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial u}{\partial x_i} + a(x_1, \dots, x_n) u = f(x_1, \dots, x_n). \tag{3, 37}$$

It turns out that the sign of the coefficient of $a(x_1, \dots, x_n)$ is of great importance as regards the possibility of solving the Dirichlet problem (first boundary-value problem) for this equation. If this coefficient is positive, then, G being sufficiently large, it may not be possible to solve the Dirichlet problem for equation (3, 37) even for the case of constant coefficients (in (3, 37)). Thus, for example, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2k^2 u = 0 \tag{4,37}$$

has the solution $u_0 = \sin kx \sin ky$ which vanishes on the boundary of the square Q with sides

$$x=0, \quad y=0, \quad x=\pi/k, \quad y=\pi/k.$$

On the other hand, it is easy to show that, if equation (4, 37) has a solution u_0 which vanishes on the boundary (supposed sufficiently smooth) of a region G and has continuous first derivatives there, then every other sufficiently smooth solution u of equation (4, 37) must satisfy on the boundary of G the condition

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† Cf. the paper of M. V. Keldish in *Uspekhi matematicheskikh nauk,* issue VIII (1941). This issue also contains a number of papers on the theory of elliptic equations and it lists relevant literature.

$$\int (\partial u_0/\partial n)\, u\, ds = 0, \tag{5,37}$$

where the integral is taken over the boundary of G. (To obtain the relation (5, 37), we must so integrate by parts the left side of the equality

$$\iint_G u_0\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2k^2u\right)dx\,dy = 0$$

that the derivatives with respect to x and y of u should not occur in the integrals taken over G.) Consequently, if G is a square and the function prescribed on the boundary of G does not satisfy the relation (5, 37), it is not possible to solve the Dirichlet problem for equation (4, 37).

When solving the Dirichlet problem for the elliptic equation (5, 37), we find that the case

$a \leqslant 0$ at all points of the considered region

differs fundamentally from the case

$a > 0$ at some points of the considered region.

In the first case the Dirichlet problem has a unique solution for a continuous function prescribed on the boundary if:

(1) The boundary of G is sufficiently smooth.

(2) The coefficients a_{ij}, a_i and the function f as well as their derivatives of sufficiently high order are continuous.

For a unique solution to exist in the second case it suffices to require, in addition, that the area of the region G be sufficiently small. In this connexion V. V. Nemytsky† showed that in this case the requirement of smallness of the area of G is also important for more general (non-linear) elliptic equations and that the magnitude of the diameter of G is irrelevant.

O. A. Oleynik considered the Dirichlet problem (*a*) for arbitrary regions G in which $a \leqslant 0$, and (*b*) for sufficiently small regions in other cases. She showed that, regardless of what continuous function is prescribed on the boundary of the region G, the conditions which must be imposed on the boundary of that region to insure the solvability of the problem do not depend on whether the Dirichlet problem is being solved for the Laplace equation or for equation (3, 37).†

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† V. V. Nemytsky, 'Existence and uniqueness theorems for non-linear integral equations, *Matematichesky sbornik*, vol. XLI (1934), pp. 438–52 (in Russian).

S. N. Bernstein proved the existence of the solution of the Dirichlet problem for a very large class of non-linear elliptic equations. A survey of these results as well as a detailed survey of other properties of elliptic equations can be found in the eighth issue of *Usepkhi matematicheskikh nauk* [*Progress of the Mathematical Sciences*]. The same volume contains a number of very important studies concerning elliptic equations.

3. All sufficiently smooth, i.e. having a sufficient number of continuous derivatives, solutions of elliptic equations and elliptic systems of equations are analytic, provided the left sides of such equations are analytic. (We suppose the right sides of these equations to be zero.) This was for the first time proved by S. N. Bernstein for elliptic equations of the second order in two independent variables.‡

4. If a homogeneous ($f \equiv 0$) elliptic equation (3, 37) with sufficiently smooth coefficients has in some bounded region G one (and, consequently, only one) solution of the Dirichlet problem for every continuous function prescribed on the boundary, then Harnack's first theorem holds for the solution functions, namely, if a sequence of such functions converges uniformly on the boundary of G, it also converges uniformly in the interior of G and the limit function satisfies equation (3, 37).

5. *An analogue of Harnack's second theorem.* Let us assume that the bounded region G is such that for every continuous function prescribed on its boundary the Dirichlet problem has one and only one solution. If the series

$$u_1(x_1, \ldots, x_n) + u_2(x_1, \ldots, x_n) + \ldots,$$

made up of non-negative (in G) solutions of the homogeneous ($f \equiv 0$) equation (3, 37), converges at least at one point of G, it converges uniformly in every closed finite region G^* contained with its boundary in G, and the limit function satisfies equation (3, 37).

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† O. A. Oleynik, 'On the Dirichlet problem for equations of the elliptic type', *Matematichesky sbornik*, vol. XXIV, 1 (1949), pp. 1–14 (in Russian).

‡ S. N. Bernstein, 'Sur la nature analytique des solutions des équations aux derivées partielles du second ordre', *Math. Ann.* vol. LIX (1904), pp. 20–76. I. G. Petrovsky, 'On the analytic nature of solutions of systems of differential equations', *Matematichesky sbornik*, vol. V (47), 1 (1939), pp. 3–70 (in Russian).

6. If in equation (3, 37) $a \equiv 0$ and $f \equiv 0$, then every solution of this equation takes on its least and largest values on the boundary of G.

7. When considered in a Riemannian space with a suitably chosen metric, the solutions of the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = 0 \qquad (6, 37)$$

have the property of the arithmetic mean. For such equations one can construct functions analogous to the (point, simple layer, and double layer) potentials for the ordinary Laplace equation in two independent variables. Analogous solutions (so-called 'fundamental solutions') were constructed by E. E. Levi for some elliptic systems (cf. footnote on p. 196).

8. Liouville's theorem for analytic functions can be extended to some elliptic equations of the second order. In this connexion S. N. Bernstein† proved the following theorem:

Every bounded, twice continuously differentiable (in the plane) solution of the equation

$$A(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, u_{xx} + 2B(\;)\, u_{xy} + C(\;)\, u_{yy} = 0,$$

where A, B, C, are bounded functions of their arguments and $AC - B^2 > 0$, is a constant.

9. If all functions u_i satisfying a certain homogeneous linear elliptic system in n independent variables with analytic coefficients vanish simultaneously together with all their derivatives of order up to $(n_i - 1)$ on a certain $(n-1)$-dimensional analytic surface, then these functions are identically equal to zero in the region in which they satisfy the system under consideration.

This theorem can be obtained, for instance, as a consequence of Holmgren's theorem on the uniqueness of the solution of the Cauchy problem for linear systems with analytic coefficients and with no real characteristics.

10. If all coefficients a_{ij} of the elliptic equation (3, 37) have bounded derivatives of order up to p in some finite region G, and if the coefficients a_i, a and f have bounded derivatives of order up to $p-1$ in this region, then all continuous solutions of this equation have bounded derivatives of order up to p in every region G^* contained, together with its boundary, in G.

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† S. N. Bernstein, 'On a certain geometric theorem and its application to elliptic partial differential equations', *Uspekhi matematicheskikh nauk*, issue VIII (1941), pp. 75–81 (in Russian).

Recently, L. K. Pogorelov showed that, if the function

$$F(x, y, z, p, q, r, s, t)$$

is continuously differentiable $k > 3$ times with respect to all its arguments, then all continuous solutions of the elliptic equation

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0$$

are differentiable at least $k + 1$ times.

11. The Dirichlet problem is a typical boundary-value problem for the Laplace equation. A typical boundary-value problem for the 'polyharmonic' equation

$$\Delta^m u \equiv \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right)^m u = 0$$

is the problem of finding a solution u of this equation in some region G given the values of u and its normal derivatives of order up to and including $m - 1$ on the boundary of G. Important problems in the theory of elasticity lead to the above boundary-value problem for $m = 2$ and $n = 2, 3$. The existence and uniqueness of the (ordinary) solution of this equation have been proved under sufficient smoothness assumptions imposed on the boundary of G and on the functions prescribed on this boundary. For $m = 2$ and $n = 2$, it suffices to require that the region G be bounded by a finite number of closed curves for each of which the coordinates are three times continuously differentiable functions of the corresponding arc length and that the functions prescribed on these curves be continuous together with their first derivatives with respect to the corresponding arc length. S. L. Sobolev showed the existence and uniqueness of the generalized solution of this problem under very general assumptions on the boundary of G. He assumed that this boundary consisted of a certain number of pieces of different dimensions. It turned out that on a piece of dimension $n - r$ one must prescribe the values of the function u and its derivatives of order up to $m - [\frac{1}{2}r] - 1$. The solution of S. L. Sobolev is a generalized solution in the sense that the function u and its derivatives need take on the prescribed values only 'in the mean' rather than at all boundary points, as was the case in the Dirichlet problem. The precise definition of the term 'in the mean' is given in the paper of S. L. Sobolev published in *Matematichesky sbornik* [*Mathematical Collection*], vol. II (44), (1937), pp. 467–500.

CHAPTER IV

PARABOLIC EQUATIONS

§ 38. Conduction of heat in a bounded strip (the first boundary-value problem)

1. *Statement of the problem.* We wish to find in the rectangle Q: $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$, a continuous function $u(t, x)$ satisfying in the interior of Q the equation

$$\partial u/\partial t = \partial^2 u/\partial x^2 \tag{1, 38}$$

and on the boundary of Q the initial condition

$$u(0, x) = \phi(x)$$

and the boundary conditions

$$u(t, 0) = f_1(t), \quad u(t, l) = f_2(t) \quad (0 \leqslant t \leqslant T).$$

The functions $\phi(x)$, $f_1(t)$, $f_2(t)$ are assumed to be continuous and $\phi(0) = f_1(0)$, $\phi(l) = f_2(0)$.

This problem can be interpreted from the point of view of physics as the problem of finding the temperature $u(t, x)$ in a strip given its initial temperature at $t = 0$ and the temperature at the edges over a subsequent period of time. The fact that the solution is sought for $t > 0$ is of fundamental importance. As will be shown later, the analogous problem for $t < 0$ does not, in general, have a solution. In distinction to the equation of the vibrating string, equation (1, 38) changes in an essential manner if t is replaced by $-t$. This is a typical equation of a non-reversible process.

The problem formulated above plays as important a role in the theory of equations of the form (1, 38) as does the Dirichlet problem in the theory of the Laplace equation. It is referred to as the *first boundary-value problem for the heat* equation (1, 38).

In the sequel we consider only continuous solutions of the heat equation.

Since equation (1, 38) does not change when t is replaced by $t + t_0$ and x by $x + x_0$, all that will be said in the sequel about the rectangle Q is true for every other rectangle with sides parallel to the coordinate axes.

2. *The maximum-minimum theorem and its consequences. Every solution u of the heat equation defined and continuous in the rectangle Q*

$$0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T,$$

takes on its least and largest values on the lower base ($t = 0$) *or on its vertical sides* ($x = 0, x = l$).

Since the minimum theorem can be reduced to the maximum theorem by changing the sign of u, we prove only the maximum part of the theorem stated above.

The method of proof is similar to that used by Privalov in proving the maximum theorem for harmonic functions. We denote by M the maximum of u in the rectangle Q and by m the maximum of the values of u on the vertical sides and on the lower base of Q. We assume that there exists a u for which $M > m$, that is, a u for which the maximum theorem does not hold. Let this function take on the value M at the point (t_0, x_0) $(t_0 > 0, 0 < x_0 < l)$. We consider the function

$$v(t, x) = u(t, x) + \frac{M - m}{4l^2}(x - x_0)^2.$$

On the vertical sides and lower base of Q

$$v(t, x) \leqslant m + \frac{M - m}{4} < \frac{M}{4} + \tfrac{3}{4}m = \theta M,$$

where $0 < \theta < 1$, and $\qquad v(t_0, x_0) = M$.

Consequently, $v(t, x)$ does not assume its maximum value on the vertical sides and lower base of Q. Let $v(t, x)$ assume its maximum value at the point (t_1, x_1) $(t_1 > 0, 0 < x < l)$. At this point we must have $\partial^2 v/\partial x^2 \leqslant 0$ and $\partial v/\partial t \geqslant 0$ (if $t_1 < T$, $\partial v/\partial t$ must vanish at this point; if $t_1 = T$, $\partial v/\partial t \geqslant 0$). We must therefore have at this point

$$\partial v/\partial t - \partial^2 v/\partial x^2 \geqslant 0. \tag{2,38}$$

On the other hand

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{M - m}{2l^2} = -\frac{M - m}{2l^2},$$

which contradicts (2, 38).

Consequences. 1°. *The first boundary-value problem has a unique solution in the rectangle Q.* For proof note that the difference of two solutions would vanish on the vertical sides and lower base of Q. Consequently, this difference would take on its maximum or minimum in Q or on its upper side, which contradicts the theorem just proved.

2°. *The solution of the boundary-value problem for the heat equation depends continuously on the functions prescribed on the vertical sides and base of Q.* This also follows from the fact that the difference of two solutions u_1 and u_2 of equation (1, 38) in the rectangle Q takes on its largest and least values on the vertical sides or on the lower base of Q.

3. *Proof of the existence of a solution of the first boundary-value problem in a rectangle.* In this proof we make use of the method of Fourier. Its chief drawback is the fact that it can be applied directly only to the case of homogeneous boundary conditions $u(t,0)=0$, $u(t,l)=0$. On the other hand, using another method, it is possible to prove the existence of the first boundary value problem for arbitrary $u(t,0)=f_1(t)$, $u(t,l)=f_2(t)$ satisfying the condition formulated in the beginning of this section.

However, if we can find any solution $v(t,x)$ of equation (1, 38) satisfying certain definite boundary conditions $v(t,0)=\tilde{f}_1(t)$, $v(t,l)=\tilde{f}_2(t)$ (nothing need be known about v as regards initial conditions), we can use Fourier's method to find a solution of equation (1, 38) for an arbitrary initial condition $u(0,x)=\phi(x)$ and for the boundary conditions $u(t,0)=\tilde{f}_1(t)$, $u(t,l)=\tilde{f}_2(t)$. For this it suffices to find by Fourier's method a solution $u^*(t,x)$ of (1, 38) satisfying the initial condition $u^*(0,x)=\phi(x)-v(0,x)$ and the homogeneous boundary conditions $u^*(t,0)=0$, $u^*(t,l)=0$. If $u^*(t,x)$ has been found, then the function

$$u(t,x)=u^*(t,x)+v(t,x)$$

yields the solution of the original problem.

In the sequel we shall need the solution of the first boundary value problem for boundary conditions of the form

$$\left.\begin{aligned} u(t,0)&=a+kt,\\ u(t,l)&=b+kt. \end{aligned}\right\} \qquad (3,38)$$

We assume that the function $\phi(x)$ is continuous, that it satisfies the conditions $\phi(0)=a$, $\phi(l)=b$, and that it has a bounded derivative. The latter requirement is again dictated not by the nature of our problem but by the method of proof used. The function

$$v(t,x)=kt+a+\frac{b-a}{l}x+\frac{x(x-l)}{2}k$$

satisfies the heat equation and the boundary conditions (3, 38). Hence, if $u^*(t,x)$ is the solution of the first boundary-value problem for equation (1, 38) for $f_1(t) \equiv f_2(t) \equiv 0$ and for the initial function

$$\phi^*(x) = \phi(x) - a - \frac{b-a}{l}x - \frac{x(x-l)}{2}k,$$

then $u = u^* + v$ is the solution of that problem for the original boundary conditions. It is clear that the function $\phi^*(x)$ is continuous, has a bounded derivative, and vanishes at $x = 0$ and $x = l$.

Just as in the case of the mixed initial and boundary-value problem for a hyperbolic equation, we seek a function $u^*(t, x)$ in the form of a series

$$u^*(t, x) = \sum_{k=1}^{\infty} T_k(t) X_k(x),$$

each of whose terms satisfies the heat equation and vanishes at $x = 0$ and $x = l$. For this we must have

$$\frac{T_k'(t)}{T_k(t)} = \frac{X_k''(x)}{X_k(x)} = -\lambda_k^2, \tag{4, 38}$$

$$X_k(0) = X_k(l) = 0. \tag{5, 38}$$

Here $-\lambda_k^2$ denote certain constants. That these constants must be negative is proved just as in § 20. From equation (4, 38) and the conditions (5, 38) we find

$$X_k(x) = A_k \cos \lambda_k x + B_k \sin \lambda_k x, \quad A_k = 0, \quad \lambda_k l = k\pi,$$

where the k are positive integers. Hence

$$X_k(x) = B_k \sin \frac{k\pi}{l} x.$$

Substituting the values of λ_k in (4, 38), we get

$$T_k(t) = B_k^{(1)} \exp\left\{-\frac{k^2\pi^2}{l^2} t\right\}.$$

Hence
$$u^*(t, x) = \sum_{k=1}^{\infty} C_k \exp\left\{-\frac{k^2\pi^2}{l^2} t\right\} \sin \frac{k\pi}{l} x. \tag{6, 38}$$

Here the C_k are constants. These constants are determined from the initial condition

$$u^*(0, x) = \phi^*(x) = \sum_{k=1}^{\infty} C_k \sin \frac{k\pi}{l} x. \tag{7, 38}$$

Since we assumed that $\phi^*(x)$ is continuous, that its derivative is bounded, and that $\phi^*(0) = \phi^*(l) = 0$, the C_k are given by the Fourier formulas. The C_k are bounded. As is known from the theory of trigonometric series, the series (7, 38) with these coefficients converges absolutely and uniformly to $\phi^*(x)$. Since for $t \geqslant 0$

$$0 < \exp\left\{-\frac{k^2\pi^2}{l^2}t\right\} \leqslant 1,$$

the series (6, 38) also converges absolutely and uniformly for $t \geqslant 0$. Consequently, the function $u^*(t, x)$ defined by this series is continuous in the rectangle Q: $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$, and takes on the prescribed values on its lower base and vertical sides. It remains to show that $u^*(t, x)$ satisfies the heat equation in the interior of Q and on its upper base. For this it suffices to show that the series obtained by differentiating (6, 38) termwise once with respect to t and twice with respect to x converge absolutely and uniformly. This follows from the fact that for every positive t

$$\frac{k^2\pi^2}{l^2}\exp\left\{-\frac{k^2\pi^2}{l^2}t\right\} < 1,$$

provided k is sufficiently large.

It is possible to show in the same way as above that the function u^* has continuous derivatives of all orders with respect to x and t in the interior of Q and on its upper base. Using this fact it is easy to show that, if we keep the original homogeneous conditions at $x = 0$ and $x = l$, it is impossible, in general, to extend the just constructed function $u^*(t, x)$ in the direction of the negative t's so that it satisfies equation (1, 38). Indeed, if such extension were possible, we would obtain a solution of the heat equation in a rectangle Q_1

$$0 < x < l,$$

$$-\epsilon < t \leqslant 0,$$

and this solution would vanish at $x = 0$ and $x = l$. Applying to Q_1 the same considerations which we applied before to Q, we find that $u^*(0, x)$, i.e. $\phi^*(x)$, must have derivatives of all orders.

Even if $u^*(0, x) = \phi^*(x)$ were such that it would be possible to solve the first boundary-value problem in Q_1 with $u^*(0, x) = \phi^*(x)$ and with homogeneous conditions at the end-points of the interval $(0, l)$, we could alter this solution by as much as we please for arbitrarily small negative values of t by changing by as little as we please the function $\phi^*(x)$ and any of its derivatives up to some arbitrary fixed order k. For this, as is easily verified, it suffices to add to the original solution a term of the series (6, 38) with sufficiently large subscript and with an arbitrarily small constant multiplier. Consequently, if the initial conditions refer to $t=0$, the first boundary value problem is reasonable for $t>0$ and is not reasonable for $t<0$ (cf. § 8). This again demonstrates the disparity between positive and negative values of t in the case of the heat equation (1, 38).

§ 39. Conduction of heat in an infinite strip (the Cauchy problem)

1. *Statement of the problem.* We wish to determine for $t \geqslant 0$ a continuous and bounded function of t and x which satisfies the heat equation (1, 38) for $t>0$ and which, for $t=0$, becomes equal to a given continuous and bounded function $\phi(x)$, defined for all real x.

2. *Proof of the uniqueness of the solution.* It is clear that to prove uniqueness it suffices to show that, for $\phi(x) \equiv 0$, the solution of the problem just stated is identically equal to zero. We denote by $u(t,x)$ some bounded solution of the latter problem whose absolute values do not exceed M, and we show that $u \equiv 0$. To this end we construct an auxiliary function $u_1(t,x)$ defined by the following conditions:

(*a*) $u_1(t,x)$ must satisfy the heat equation in the interior of the square

$$|x| \leqslant 1,$$

$$0 \leqslant t \leqslant 1;$$

(*b*) $u_1(t,x)$ must be continuous in this square and satisfy the following conditions on its boundary:

$$u_1(t,0)=M, \quad u_1(t,l)=M \quad \text{for} \quad 0 \leqslant t \leqslant 1,$$

$$u_1(0,x)=0 \quad \text{for} \quad |x| \leqslant 1-\epsilon,\ \epsilon>0;$$

(*c*) for $t=0$, $u_1(t,x)$ must be linear in the intervals

$$(-1,-1+\epsilon), \quad (1-.\epsilon,1).$$

The existence of such function was proved in the preceding section. It follows from the maximum theorem that $u_1(t,x) \geqslant 0$ everywhere.

Let us put

$$u_l(t,x)=u_1\left(\frac{t}{l^2},\frac{x}{l}\right).$$

It is clear that for every positive l the function $u_l(t,x)$ satisfies the heat equation in the interior of the rectangle

$$|x| \leqslant l,$$
$$0 \leqslant t \leqslant l^2,$$

that it is equal to M on its vertical sides, that it vanishes on the segment $t=0$, $|x| \leqslant l(1-\epsilon)$, and that $u_l(0,x)$ is linear in the intervals $(-l, -l(1-\epsilon))$ and $(l(1-\epsilon), l)$. Since the function $u_1(t,x)$ vanishes at the point $t=x=0$ and is continuous in its neighborhood, all values of $u_1(t/l^2, x/l)$, l sufficiently large, are positive and arbitrarily small in any fixed rectangle Q^*,

$$|x| \leqslant X;$$
$$0 \leqslant t \leqslant T,$$

On the other hand, it is easily verified that, for arbitrary sufficiently large l, a solution $u(t,x)$ of our problem satisfies the inequality

$$|u(t,x)| \leqslant u_l(t,x) \tag{1,39}$$

on the lower base and vertical sides of the rectangle Q^*. Consequently, the same inequality must hold in the interior of Q^*. However, according to what has just been said, if (t,x) is an arbitrary but fixed point of the square Q^* with $t>0$, the right side of the inequality (1, 39) is arbitrarily small for l sufficiently large, and the left side of (1, 39) is independent of l. Hence $u(t,x)=0$ at an arbitrary point (t,x), $t>0$. This completes the proof of uniqueness.

A. N. Tikhonov showed that the solution of the problem considered may not be unique in the class of unbounded functions and found a uniqueness condition.†

3. *Continuous dependence of the bounded solution on the initial function* $\phi(x)$. To prove this continuous dependence it suffices to show that $|\phi(x)| \leqslant \epsilon$ implies $|u(t,x)| \leqslant \epsilon$. This is proved in a manner analogous to that of the uniqueness proof just given, with the auxiliary function $u_1(t,x)$ replaced by the function $u_1(t,x)+\epsilon$.

4. We prove that the solution of our problem is given by the formulas

$$u(t,x) = \frac{1}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\phi(\xi)}{t^{\frac{1}{2}}} \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi \quad \text{for} \quad t>0, \tag{2,39}$$

$$u(0,x) = \phi(x). \tag{3,39}$$

The integral (2, 39) is known as the *Poisson integral.*

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† *Matematichesky sbornik*, vol. XLII, 2 (1935), pp. 199–216 (in Russian).

It is easy to check that the integral (2, 39) converges for all positive t. It is just as easy to show that the integrals obtained by differentiating (2, 39) under the integral sign any number of times with respect to t and x converge. Also, all these integrals converge uniformly in the neighborhood of (t, x) $(t > 0)$. It follows that, for $t > 0$, $u(t, x)$, defined by (2, 39), and its derivatives of all orders with respect to x and t exist. Since the integrand satisfies equation (1, 38) for $t > 0$, it follows that $u(t, x)$ itself satisfies equation (1, 38) for $t > 0$.

We now show that the function $u(t, x)$ defined by formula (2, 39) is bounded for $t > 0$. We first observe that, if we put

$$M_1 = \max_{-\infty < x < \infty} |\phi(x)|,$$

then

$$|u(t,x)| \leqslant \frac{1}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{M_1}{t^{\frac{1}{2}}} \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi = \frac{M_1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\{-\eta^2\}\, d\eta = M_1.$$

It remains to show that the function $u(t, x)$ is continuous for $t = 0$, i.e. for every x_0

$$\left| \frac{1}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\phi(\xi)}{t^{\frac{1}{2}}} \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi - \phi(x_0) \right| < \epsilon, \tag{4, 39}$$

provided t and $|x - x_0|$ are sufficiently small. We observe that to prove the continuity of $u(t, x)$ for $t = 0$ it suffices to prove that for small t

$$\left| \frac{1}{2(\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} d\xi - \phi(x) \right| < \tfrac{1}{2}\epsilon. \tag{5, 39}$$

That this is so follows from the fact that, in view of the continuity of the function $\phi(x)$, $|\phi(x) - \phi(x_0)|$ is small if $|x - x_0|$ is sufficiently small.

To prove the relation (5, 39) we rewrite the Poisson equation in the form

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(x + 2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\}\, d\zeta$$

by putting $\xi = x + 2t^{\frac{1}{2}}\zeta$, and observe that

$$\phi(x) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(x) \exp\{-\zeta^2\}\, d\zeta.$$

In view of the boundedness of $\phi(x)$, the absolute values of the integrals

$$\int_{-\infty}^{-N} \phi(x + 2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\}\, d\zeta, \qquad \int_{-\infty}^{-N} \phi(x) \exp\{-\zeta^2\}\, d\zeta,$$

$$\int_{N}^{\infty} \phi(x + 2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\}\, d\zeta, \qquad \int_{N}^{\infty} \phi(x) \exp\{-\zeta^2\}\, d\zeta$$

are as small as we please for N sufficiently large. Consequently, for N sufficiently large, we have, approximately,

$$u(t,x) = \pi^{-\frac{1}{2}} \int_{-N}^{N} \phi(x + 2t^{\frac{1}{2}}\zeta) \exp\{-\zeta^2\} d\zeta,$$

and
$$\phi(x) = \pi^{-\frac{1}{2}} \int_{-N}^{N} \phi(x) \exp\{-\zeta^2\} d\zeta,$$

and the approximation is as good as we please. But for t sufficiently small, the right sides of these approximate equalities are, by the continuity of $\phi(x)$, arbitrarily close to each other. This implies the correctness of (5, 39).

5. We have thus shown that the unique solution of the problem stated in the beginning of this section is that defined by equations (2, 39) and (3, 39).

These formulas imply in particular that, if $\phi(x)$ is positive on an arbitrarily small x interval and otherwise equal to zero everywhere, then $u(t,x)$ is positive for all values of x and for an arbitrary fixed $t > 0$. This implies the paradoxical result that heat is conducted in a strip with infinite speed. While this is physically impossible, it nevertheless follows inevitably from the assumption that equation (1, 38) provides an exact description of the conduction of heat in a strip.

Clearly, the hypotheses on which our derivation of this equation is based are not quite justified by experience.

§ 40. Survey of some further investigations of equations of the parabolic type

1. Existence and uniqueness of the solution of the first boundary-value problem for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} \tag{1, 40}$$

have been proved for arbitrary n. In the simplest case this problem can be stated as follows:

We seek a continuous function $u(t, x_1, \dots, x_n)$ defined on the closure of a region G bounded ('from below' and 'from above') by pieces of the planes $t = 0$ and $t = T$ and on the sides by one or a few surfaces with continuously turning tangent planes nowhere parallel to the t-axis. The function u must satisfy equation (1, 40) in the interior of G and must coincide with some function f, prescribed on the boundary of G with the exception of its upper base. This function is supposed continuous on the closed set on which it is defined. The proofs of the uniqueness of the solution of this problem and of its continuous dependence on the function f are carried out in exactly the same way as in § 38.

2. Results analogous to those stated in para. 1 have been proved for the case when the condition $u = f$ on the boundary is replaced by the condition $(\partial u/\partial n) + \sigma u = f$ on that boundary.

3. The results stated in paras. 1 and 2 have also been proved for the solutions of the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} A_{ij}(t, x_1, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(t, x_1, \ldots, x_n) \frac{\partial u}{\partial x_i} + C(t, x_1, \ldots, x_n) u + D(t, x_1, \ldots, x_n). \quad (2, 40)$$

Here it is assumed that the form

$$\sum_{i,j=1}^{n} A_{ij}(t, x_1, \ldots, x_n) \alpha_i \alpha_j$$

is positive-definite at every point $(t, x_1, \ldots, x_n)$ of the region under consideration, and that the coefficients A, B, C, D have continuous derivatives of sufficiently high orders.†

4. If the coefficients of equation (2, 40) are analytic in all their arguments, then, as was shown by Gevrey,† all sufficiently smooth solutions of this equation are analytic in the arguments $(x_1, \ldots, x_n)$ and have derivatives of all orders with respect to t, without being necessarily analytic in t.

5. If the initial values of u are prescribed in the whole hyperplane $t = 0$, then there exists a unique bounded solution of the Cauchy problem for equation (2, 40) (this problem is analogous to that considered in § 39). Here it is assumed that $u(0, x_1, \ldots, x_n)$ is bounded and continuous.

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† Gevrey, *J. Math.*, 6 série, vol. IX (1913), vol. X (1914); *Ann. Ecole Norm. sup.*, 3 série, vol. XXXV (1918).